# Orderable groups

A group $G$ is orderable if its elements can be put into an order ~~in a w~~ which is preserved by group multiplication. This is an extra requirement to ask of a group, because not all gps can be orderable.

In this talk, we will

- introduce the different notions of orderability
- give some important examples of orderable gps
- consider the properties of the class of orderable gps

(& motivate the study of orders on gps)

Ultimately we will prove a thm of Vinogradov:

a free product of left-orderable gps is left orderable.

Ref^s:

E. Ghys, gps acting on the circle 2001

~~R~~ Mura, Rhemtulla, ordrable gps 1977

Navas, On the dynamics of (left) orderable gps 2010

Groups, Orders, Dynamics

**Def** Total order $\leq$ on a set $X$: a binary relation satisfying

**Anti-Symmetric** $\quad x \leq y \;\&\; y \leq x \implies x = y. \quad \forall x,y \in X$

~~Reflexive~~ **Transitive** $\quad x \leq y \;\&\; y \leq z \implies x \leq z \quad \forall_{x,y,z} \in X$

**Total** $\quad x \leq y$ or $y \leq x \quad \forall x,y \in X$

(any two elmts are comparable)

(total $\Rightarrow$ reflexive, $x \leq x$. Pd: just reflexive)

**Note** total means that $x \leq x$ when $x = y$, so total orders are reflexive.

Also called "linear orders" b/c think of $X$ as laid out in a line

$\cdots \; x < y < z \; \cdots$

[Order type, ordinals]

ordinals $\leftrightarrow$ well-orderings

(e.g. $\leftarrow$ not ~~total~~ well; well: total + $\exists$ min of every set)

**Examples**

- $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ with usual $<$
- dictionary order on $A^n = \{$~~finite~~ length $n$ strings over $A\}$ (free monoid)

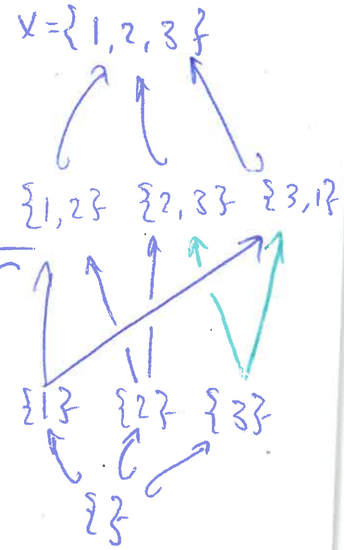
$x_1 \cdots x_n < y_1 \cdots y_m$

~~iff $x_1 \leq y_1$ or ($x_1 = y_1$ and $x_2$~~

iff $(x_1 < y_1)$
or $(x_1 = y_1 \;\&\; x_2 < y_2)$
or $(x_1x_2 = y_1y_2 \;\&\; x_3 < y_3)$
or
or ~~$(x_1 \cdots x_{l-1} = y_1 \cdots y_{l-1} \;\&\; x_l \leq y_l$, $l = \min(n,m)$~~
$(x_1 \cdots x_{n-1} = y_1 \cdots y_{n-1} \;\&\; x_n < y_n)$

**Non example**: $\mathcal{P}(X)$ order $\subseteq$ (not total)

pick disjoint sets




**Thm**: {orderings} ffe or $\geq$ Cantor

**Def** A gp $G$ is left-orderable / right-orderable / bi-orderable if $\exists$ a total order $\leq$ on $G$

Such that
$$\begin{cases} x \leq y \implies gx \leq gy \\ x \leq y \implies xg \leq yg \\ x \leq y \implies gxh \leq gyh \end{cases} \quad \forall x, y, g, h \in G$$

**Ex**, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ under addition.

Ab: LO iff BO

**Non example** $(\mathbb{R} \setminus \{0\}, \times)$ $\quad -1 \leq 1$

$\Rightarrow -1 \times -1 \leq -1 \times 1$

$\Rightarrow 1 \leq -1$

$\Rightarrow 1 = -1$ *.

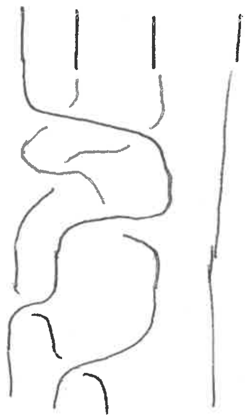
- any gp with Torsion ($x \neq id$, $x^n = id$)

**Ex:**
- Braid gps (Dehornoy) — LO

Pure : BO

Full : LO

- Diagram gps, inc Thompson's $F$ (Guba / Sapir) BO



**Why?**
- Connections to topology. Space $\longrightarrow \pi_1$ (Space)
- Interesting!
- Measures how resiliant / many relations a gp has?
- Another invariant
- Orders $\overset{\exists}{\longleftrightarrow}$ certain actions

Full : no restrictions

Pure Braid gp:

strand 1 2 3 4

1 2 3 4

(pure is subgp)

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- Orderable $\supseteq$ torsion-free nilpotent gps

free gps

Braid gps

~~$[G, G_{i+1}]$~~ $\leq G_i$

Nilpotent ~~$\{1\} \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$~~

the lower central series $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{1\}$

$G_{i+1} = [G_i, G]$

## Basics

• LO iff RO ( different orders? )

given $\leq_L$ left order define $\leq_{\text{Inv}}$ by $x \leq_R y$ iff $x^{-1} \leq_L y^{-1}$.

inverse order

$$
\begin{aligned}
xg \leq_R yg \quad & \text{iff } (xg)^{-1} \leq_L (yg)^{-1} \\
& \text{iff } g^{-1}x^{-1} \leq_L g^{-1}y^{-1} \\
& \text{iff } x^{-1} \leq_L y^{-1} \\
& \text{iff } x \leq_R y.
\end{aligned}
$$

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Apply to $\mathbb{Z}$ : $-2 \leq_L 3 \quad \longleftrightarrow \quad 2 \leq_R -3$ reverse order

$$\cdots \; -3 \;\underset{\geq_R}{\overset{\leq_L}{\text{———}}}\; -2 \;\underset{\geq_R}{\overset{\leq}{\text{———}}}\; -1 \;\underset{\geq_R}{\overset{\leq_L}{\text{———}}}\; 0 \;\underset{\geq_R}{\overset{\leq_L}{\text{———}}}\; 1 \;\underset{\geq_R}{\overset{\leq_L}{\text{———}}}\; 2 \;\underset{\geq_R}{\overset{\leq_L}{\text{———}}}\; 3 \; \cdots$$

reverse ordering!

$x \leq_{\text{Rev}} y \leftrightarrow y \leq x$

Note that $g \mapsto g^{-1}$ is an iso$^m$ $G \longrightarrow G^{\text{opp}}$

$x \circ y \mapsto y \,\square\, x$

(or an "antiautomorphism" $f(xy) = f(y)f(x)$ )

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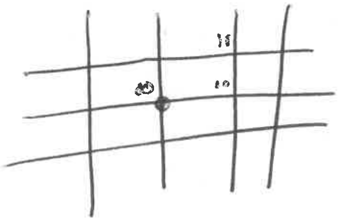
Misleading example: $\mathbb{Z}$ is Abelian, so any LO is a BO.

This shows that $\mathbb{Z}$ has $\geq 2$ orderings.

In fact, $\mathbb{Z}$ has $=2$ orderings. (determined by $1 \gtrless 0$ )

Ab: reverse order $\Updownarrow$ inverse order

How may orderings? $\mathbb{Z}^2$



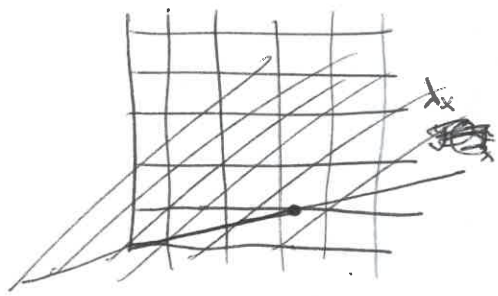
• dictionary order: $(x,y) \le (x',y') \iff \begin{cases} x \le x' & \text{or} \\ x = x' \text{ and } y \le y' \end{cases}$

BO? $(x+a, y+b) \le (x'+a, y'+b) \iff \begin{cases} x+a \le x'+a & \text{or} \\ x+a = x'+a \text{ and } y+b \le y'+b \end{cases}$

Irrational type : $\lambda$ irrational.

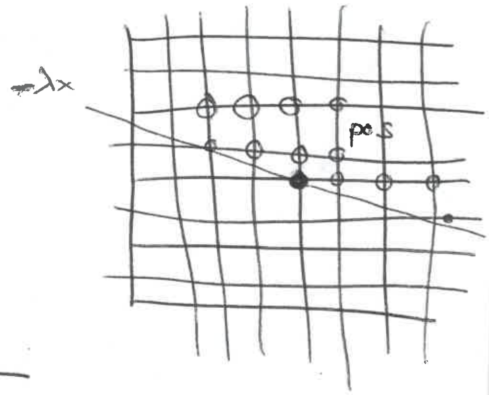
$(x,y) > (0,0)$ iff $\lambda x + y > 0$ $\qquad \lambda x + y = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \cdot (x,y)$

e.g.




More "rational types"
(I think lex is one of these)

$y + \lambda x > 0$
iff $y > -\lambda x$

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~~Prop. Let $G$ be countable. $G$ is LO $\iff$ $G$ embeds in $\mathrm{Homeo}^+(\mathbb{R})$.~~

$\exists$ topology on {orderings of $G$}.

Thm $G$ countable

{orderings} = fte $\cup$ Cantor sets

small perturbations of orderings?

**Prop** $G$ : countable

$G$ LO $\iff$ $G \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$
$\qquad\qquad\qquad\qquad \circlearrowright$
$\qquad\qquad\qquad\qquad F$ :)

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Other orderables:

- Lattice - orderable : partial, not total order.
  can define ~~[illegible]~~

$$\leq f \wedge g$$
$$f \qquad\qquad g$$
$$\geq \qquad \leq$$
$$f \vee g$$

- Circular order : ternary relation defines circular paths, preserved by mult.
  Can have torsion now!

$G$ CO $\iff$ $G \hookrightarrow \mathrm{Homeo}^+(S^1)$
$\qquad\qquad\qquad\qquad \circlearrowright$
$\qquad\qquad\qquad\qquad T$ :)

• Class of LO gps: closed under

— subgps

— products (lex; need care for infinite products) (cat prod)

— extensions $K \hookrightarrow E \xrightarrow{\pi} Q \to 1$ $(K, Q \text{ LO}) \Rightarrow E \text{ LO}$ lex order

(if $K \trianglelefteq_{LO} E$ & $E/K$ LO)

$$g \le g' \iff \begin{cases} \pi(g) <_Q \pi(g') & \text{or} \\ \pi(g) =_Q \pi(g') & \text{and} \quad g'^{-1} g \le_K 1 \end{cases}$$

(c.f. lex.)

Ex $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle x, y \mid x^2 = y^2 = 1 \rangle$

inf dihedral gp $\cong \mathbb{Z}_2 \ltimes \mathbb{Z}$

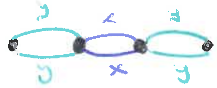
— free products (cat coprod)

$A * B = \langle X \cup Y \mid R \cup S \rangle$

$A = \langle X \mid R \rangle$ $\quad X, Y$ disjoint

$B = \langle Y \mid S \rangle$

$\left( \langle \bigcup X_i \mid \bigcup R_i \rangle \right)$




**Thm** (Vinogradov)

$G_i$ a family of LO gps. $i \in I$ (countable at least)

$\mathop{*}_{i \in I} G_i$ LO $\iff$ all $G_i$ LO

Pf. $\Rightarrow$: easy, restrict orders (cos $G_i \hookrightarrow * G_j$)

Set $P$ = free prod.

$\Leftarrow$: harder.

Define a graph $X$ with  Set $G_0 = \{1\}$.

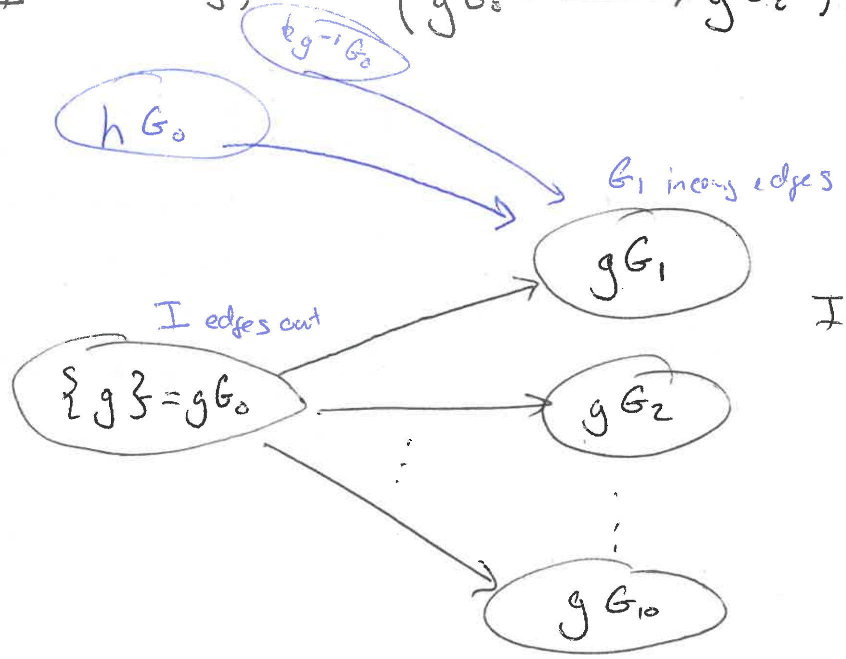
vertex set $G \times (I \cup \{0\})$ $(g, i) \leftrightarrow gG_i$

edges set $G \times I$ $(g, i) \leftrightarrow (gG_0 \longrightarrow gG_i)$

Bass–Serre theory

graphs (complexes) of gps

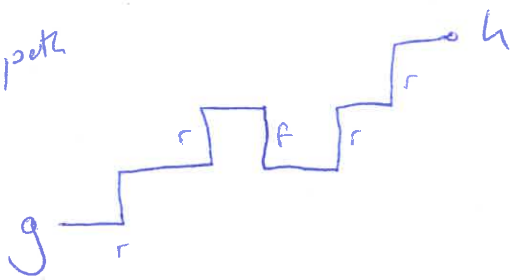
Universal Covering Tree




$I = \{1, \dots$

maybe $gG_1 = hG_1$ some $h \in G$

- $X$ is a tree
- $\exists!$ minimal path $gG_0 \longrightarrow hG_0$ any $g, g' \in G$
- at each internal vertex $\longrightarrow \circ \longrightarrow$ of this path we can compare orders
- either next edge is bigger (rise) or smaller (fall)
- count $m(g,h) =$ # rises $-$ # falls on this path
- $g \leq_{\text{free prod}} h$ if $m(g,h) > 0$




(need total order on $I$)  - check this is a LO.