# Conjugacy in Higman-Thompson groups

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2nd July, 2015

With hyperlinks!

# The plan

- Introduce these groups & their friends
- Higman's solution to the conjugacy problem
- Implementation and future work

# A recipe for Thompson's group $F$

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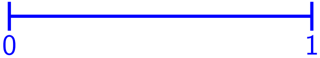

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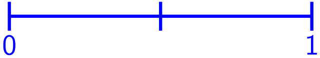

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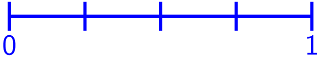

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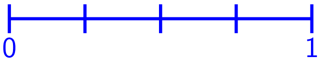

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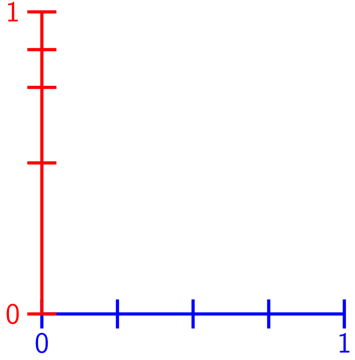

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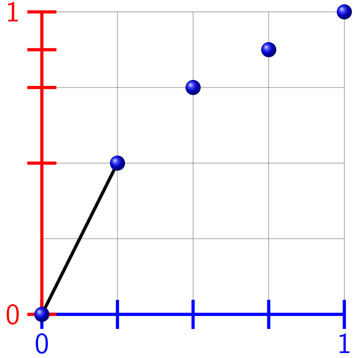

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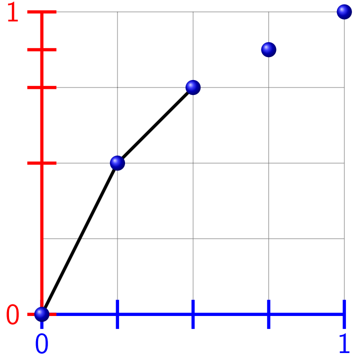

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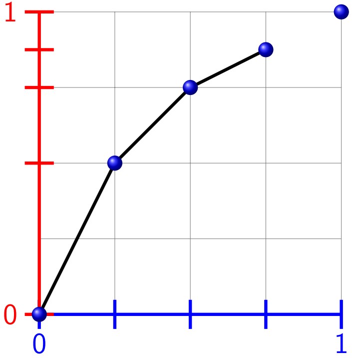

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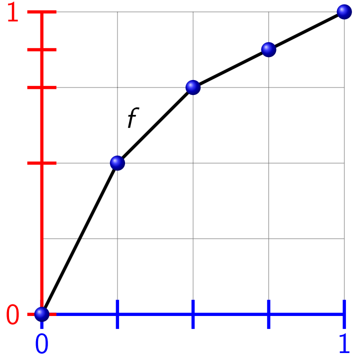
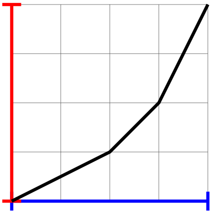


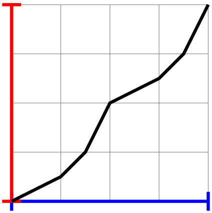

Functions $f$ like this are the elements of Thompson's group $F$.

# Example elements of $F$

$F$ is the group of all such functions under function composition.
Some more examples:



$f^{-1}$



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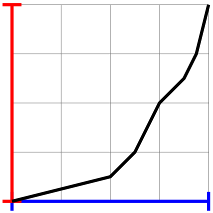

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# People *really* like $F$

- Provides interesting examples & counter-examples
- Easy to describe:

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For more details, see Jim Belk's thesis; Cannon, Floyd & Parry; or John Meier's presentation.

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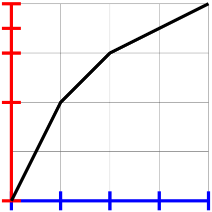

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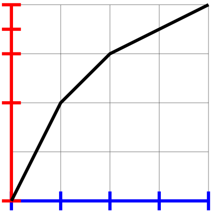

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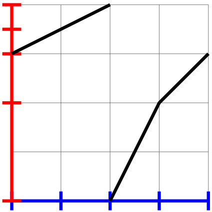

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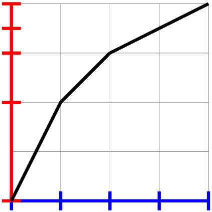

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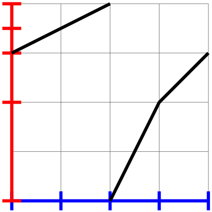

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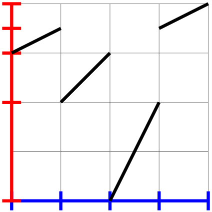

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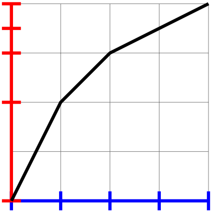

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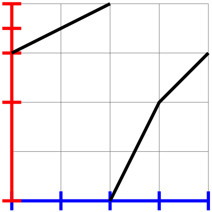

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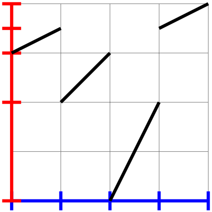

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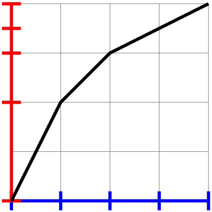

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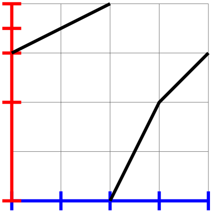

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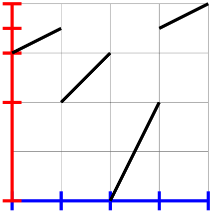

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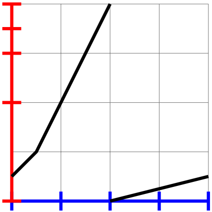

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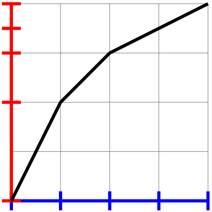

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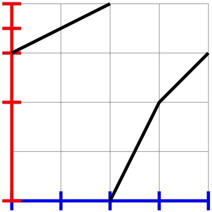

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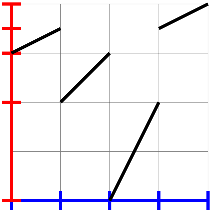

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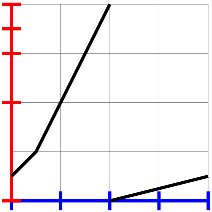

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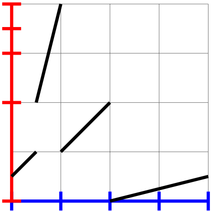

order 3 in $V$

# $F$'s friends are interesting too

$V$ contains a copy of every finite group $G$:

- $G \hookrightarrow \mathcal{S}_n \hookrightarrow \mathcal{S}_{2^m}$
- Chop interval into halves, quarters, eights, ..., $2^m$ths.
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More exotic friends lead to more f.p. inf. simple groups:

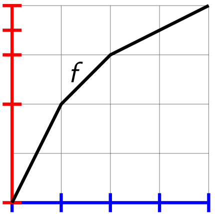
- Brin: $2V, 3V, \ldots, nV, \ldots$
- Repeatedly halve $[0,1]^2$, $[0,1]^3$, etc. and rearrange
- Higman: $F_{n,r}$, $T_{n,r}$, $V_{n,r}$
- Repeatedly chop $[0,1]$ into $n$ pieces and rearrange

# Conjugacy in $V$

- Many authors with many approaches to conjugacy in $V$:
  - 1974 Higman: automorphisms of an algebra
  - 2007 Belk, Matucci: strand diagrams
  - 2010 Salazar-Díaz: revealing tree pairs
  - 2011 Bleak *et al.*: train tracks and flow graphs
- More approaches for $F$ and $T$ & friends

I'll try to explain Higman's approach, and comment on how we got a computer to implement it.

# Tree pair diagrams

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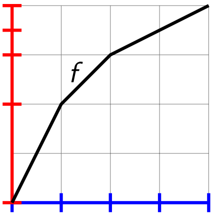


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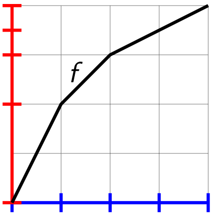


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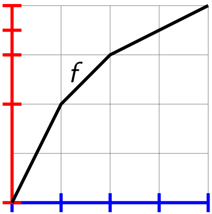


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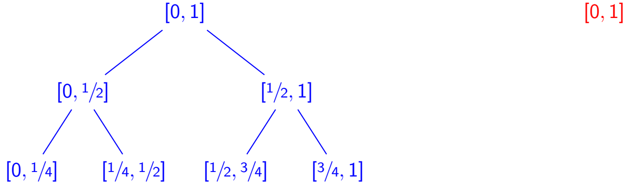

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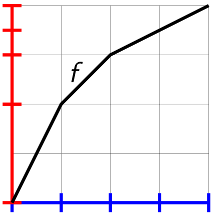


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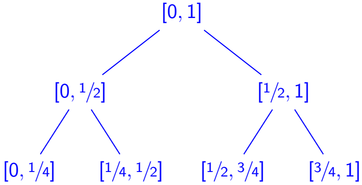


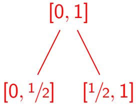

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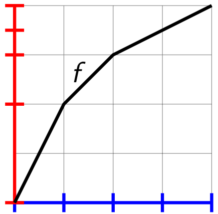


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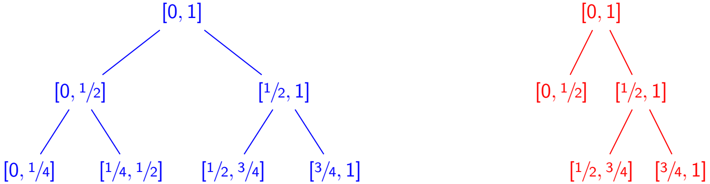

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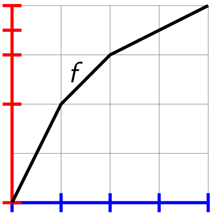


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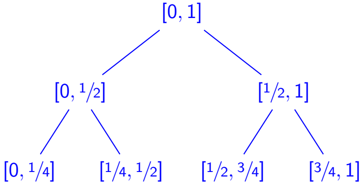


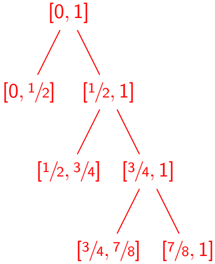


In $T$ and $V$, number the leaves to show how intervals get mapped.

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Higman described paths in the tree using an algebra. To write them down as words, he introduced some labels:

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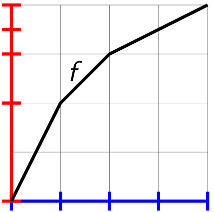


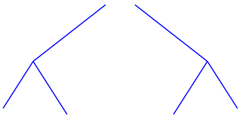

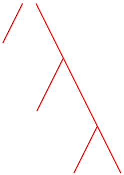

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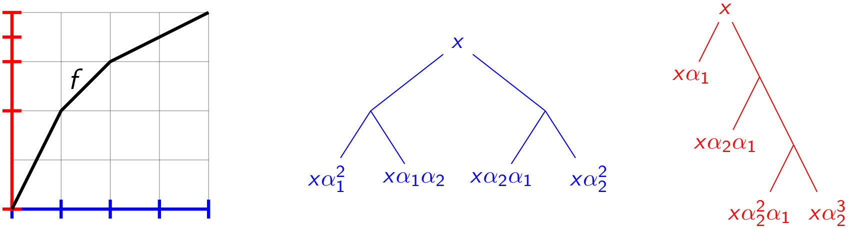

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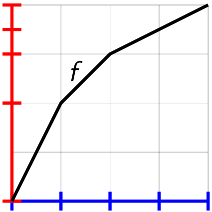
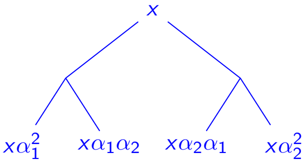
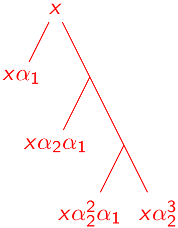


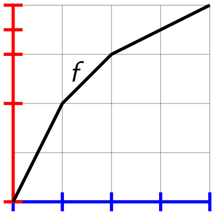




The function $f$ is completely specified by a mapping from domain to range words.

$$
\begin{aligned}
x\alpha_1^2 &\mapsto x\alpha_1 & \qquad x\alpha_2\alpha_1 &\mapsto x\alpha_2^2\alpha_1 \\
x\alpha_1\alpha_2 &\mapsto x\alpha_2\alpha_1 & \qquad x\alpha_2^2 &\mapsto x\alpha_2^3
\end{aligned}
$$

# Trees → paths → words




Can track repeated application of $f$, e.g. using $x\alpha_2^2 \mapsto x\alpha_2^3$.

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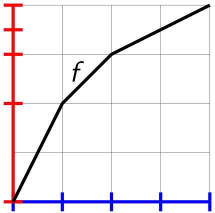


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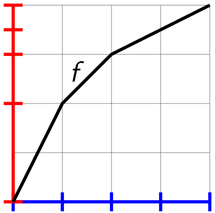


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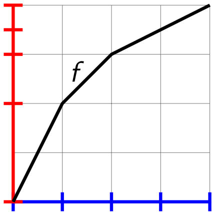


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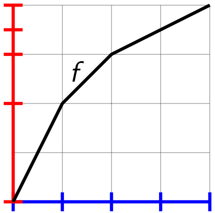


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$$\begin{aligned} [^1\!/\!_4, 1] &= [^1\!/\!_4, ^1\!/\!_2] \cup [^1\!/\!_2, 1] \\ &\approx x\alpha_1\alpha_2 \cup x\alpha_2 \\ &\approx (x\alpha_1\alpha_2)(x\alpha_2)\lambda \end{aligned}$$

# Components ($\approx$ orbits)

- Pick your favourite element/interval e.g. $w = x\alpha_1^2 \approx [0, 1/4]$.
- Compute (pre)images of $w$ until you can't any more:

$$\dots,\ f^{-2}(w),\ f^{-1}(w),\ w,\ f(w),\ f^2(w),\ \dots$$

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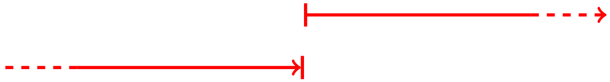

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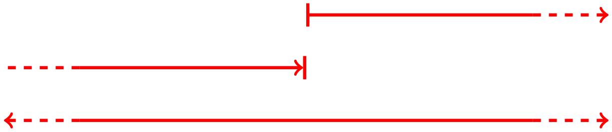

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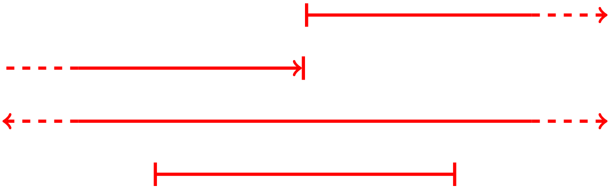

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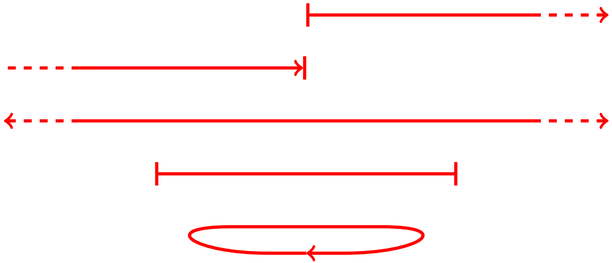

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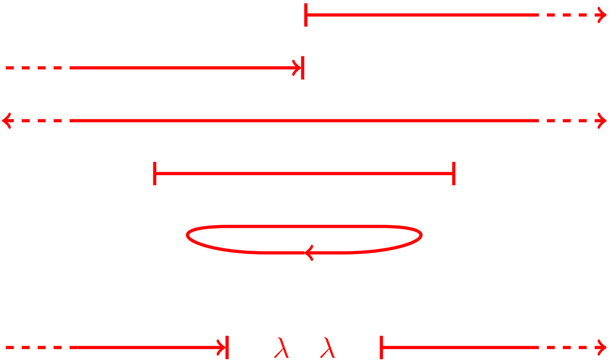

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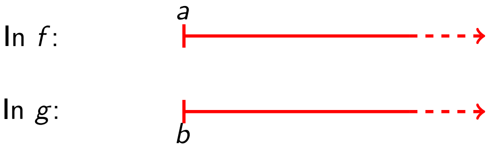

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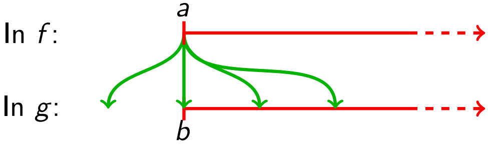

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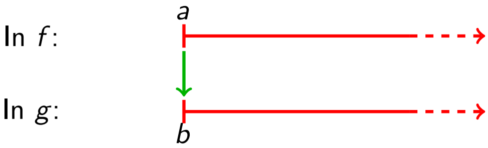

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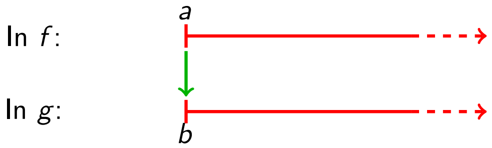


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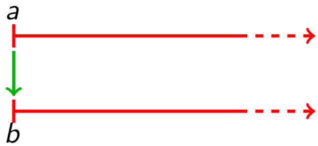


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- Some shortcuts e.g. conjugator has to preserve relations between components

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- Summer project that was more like a semester project...
- Other tools exist to do calculations in $V$, but not to solve the conjugacy problem.

Code is on GitHub and documentation online. Details written on the arXiv [with Nathan Barker & Andrew Duncan].
Time for a quick demo?

# Future Work

- Complexity?
- Simultaneous conjugacy?
  Given $x_1, \ldots, x_n; y_1, \ldots, y_n$ find a single conjugator $z$

$$z^{-1} x_i z = y_i, \quad \forall i$$

- Can we solve different kinds of equations in $V$ (and $V_{n,r}$)?
- Can we apply this to other Thompson-like groups?