Conjugacy in Higman-Thompson groups

David Robertson

15th April, 2015

With special bonus hyperlinks!

Crash course on groups

- Group: set with a nice binary operation
  - $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ with addition
  - non zero reals, invertible matrices with multiplication
  - invertible maps $f : X \to X$ with function composition

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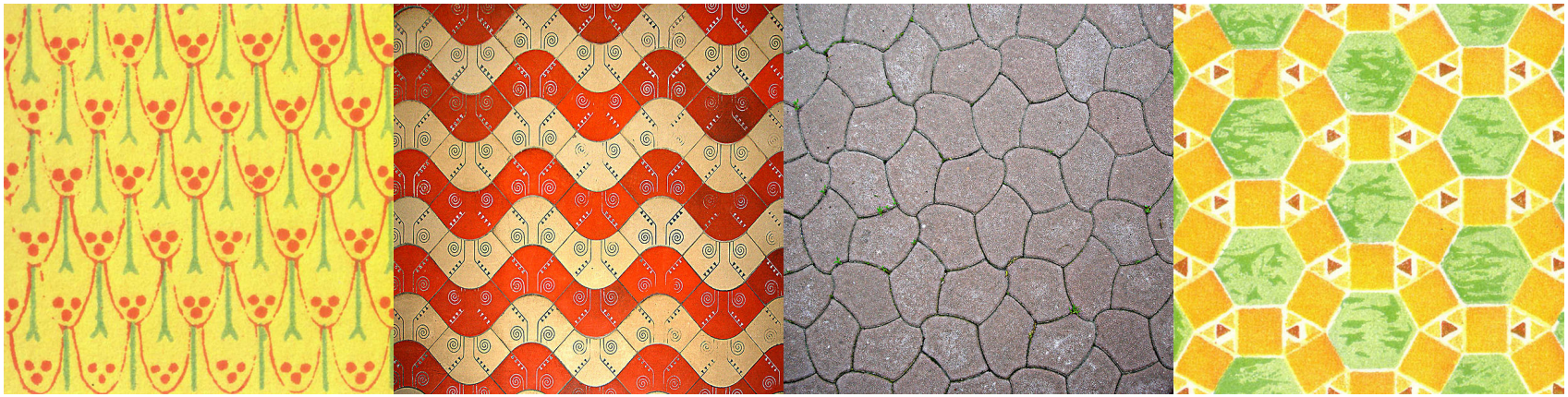

Images from Wikipedia

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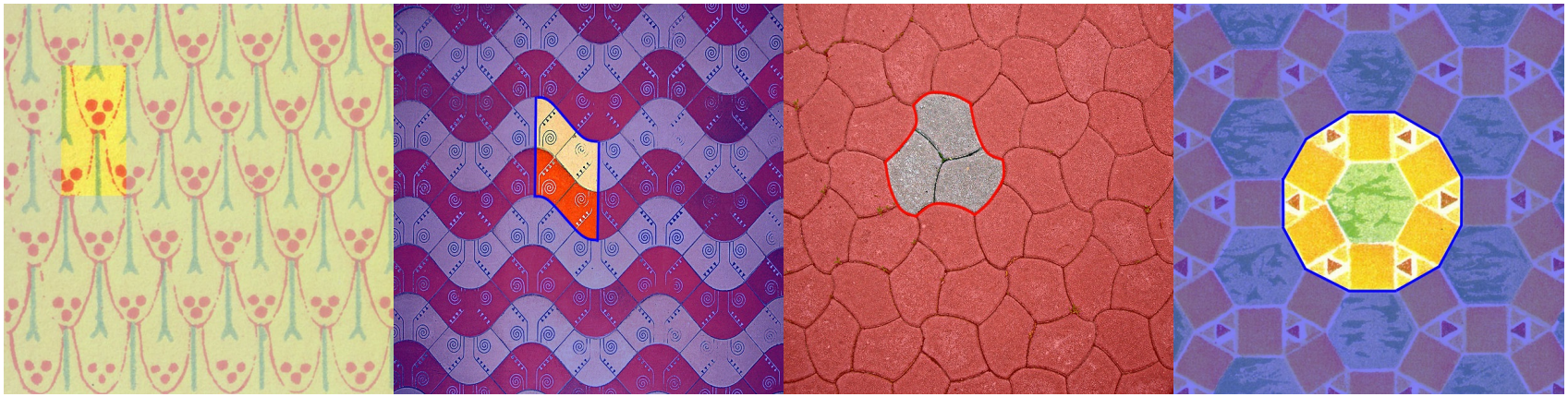

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The conjugacy problem

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- Cryptosystem?
- Can do this for specific groups, *i.e.* with G fixed.

A recipe for F

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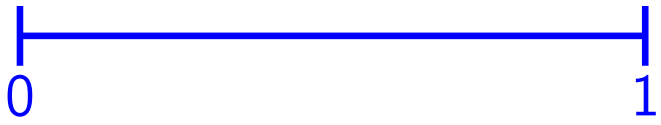

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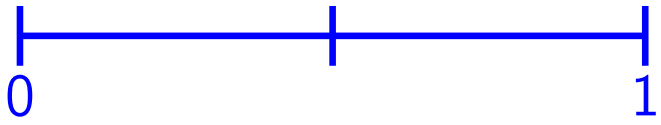

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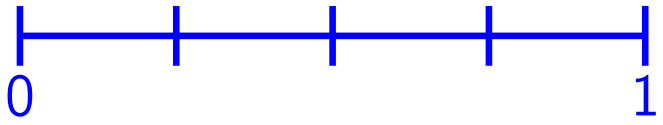

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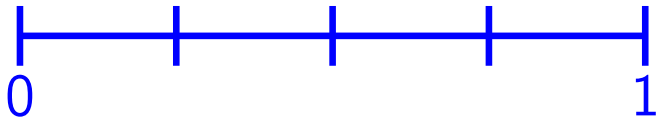

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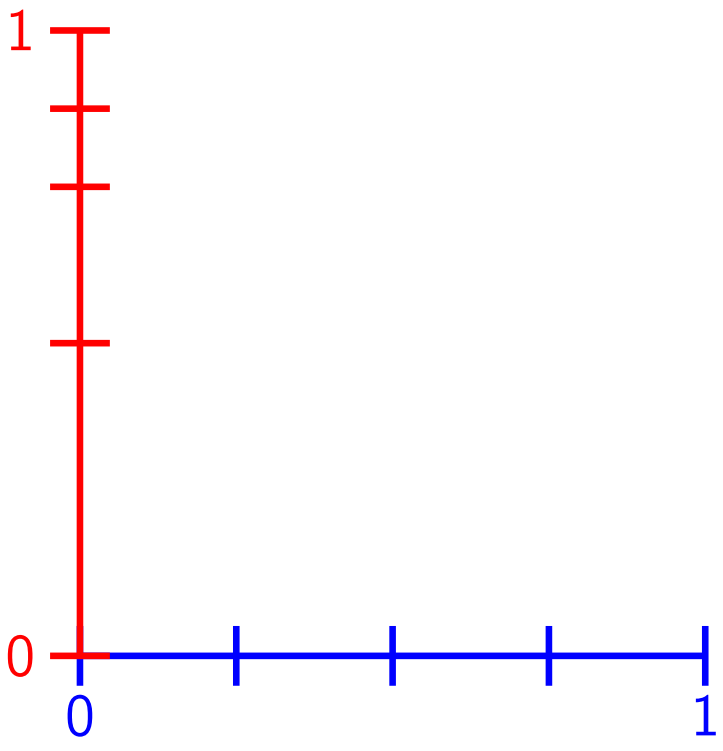

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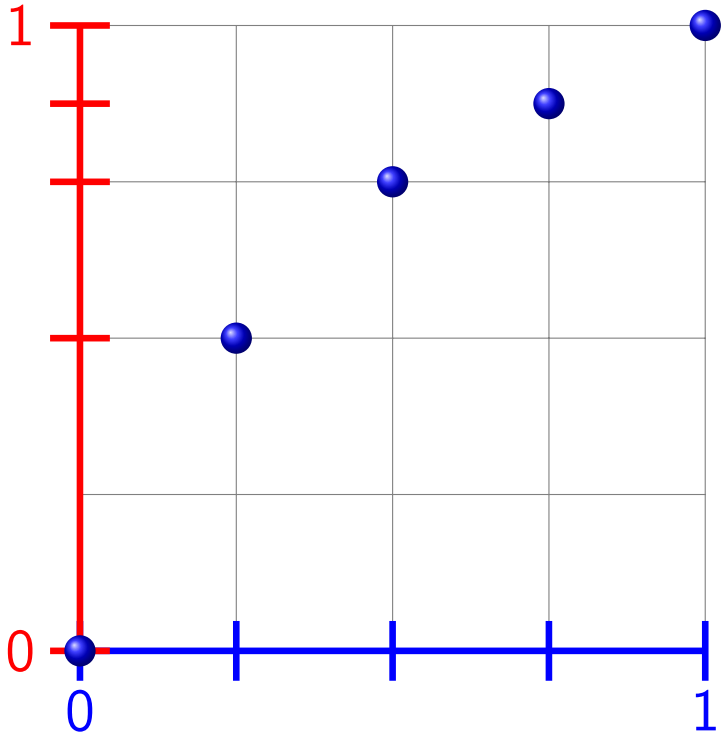

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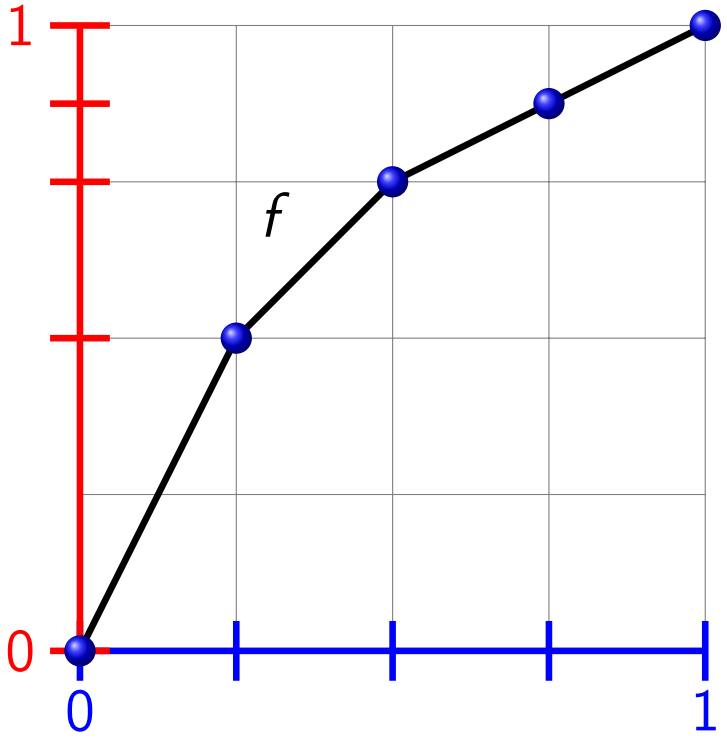
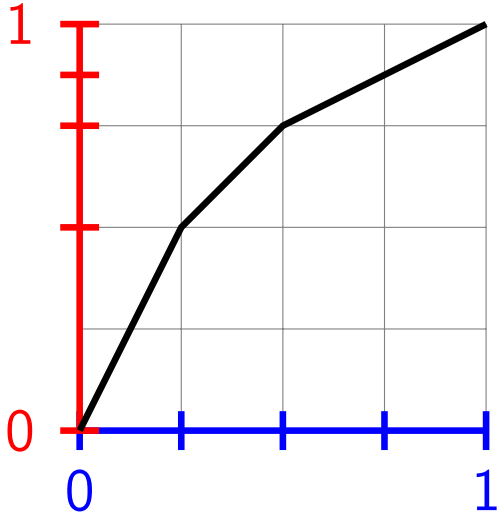


Functions f like this are the elements of Thompson's group F.

Thompson's other groups T and V

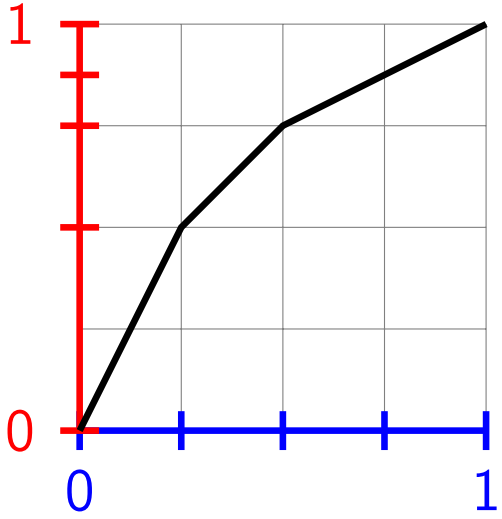
F



F: increasing functions

Thompson's other groups T and V

$$F < T$$



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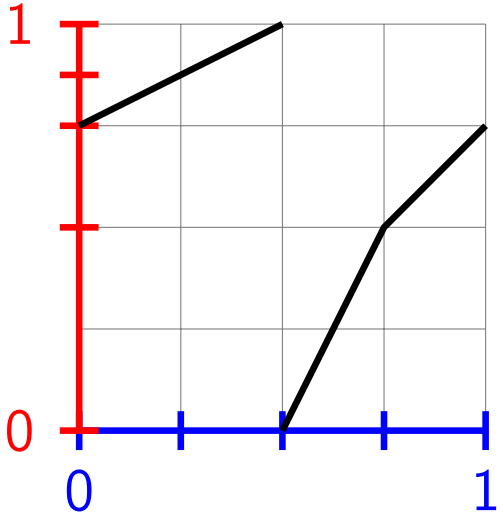

T: don't have to start at $(0, 0)$

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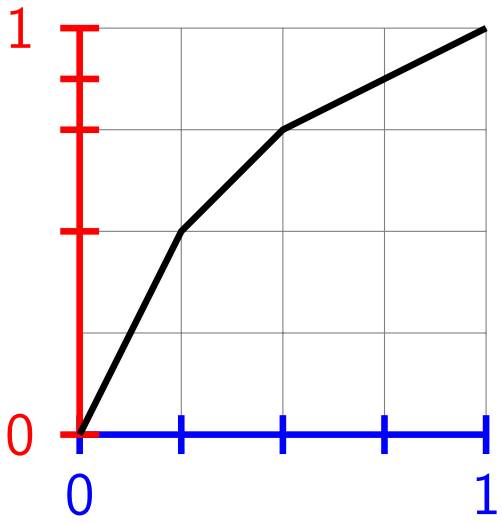

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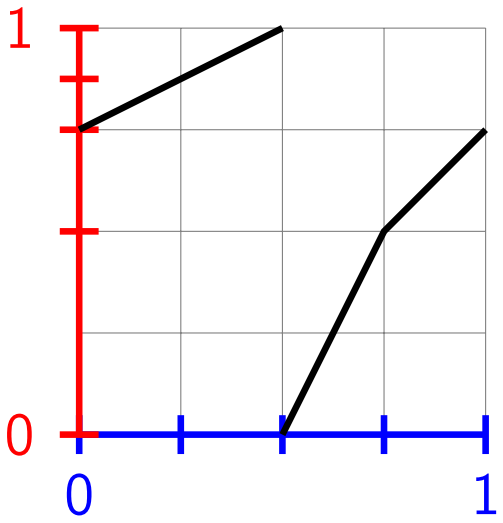

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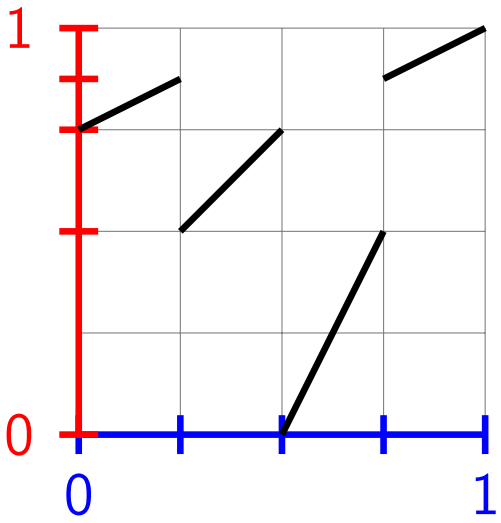

V: don't have to be continuous

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T and V are finitely presented, infinite simple groups (rare!)

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- $G \hookrightarrow \mathcal{S}_n \hookrightarrow \mathcal{S}_{2^m}$
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People are also interested in whether F is amenable or not,

whatever that means. . .

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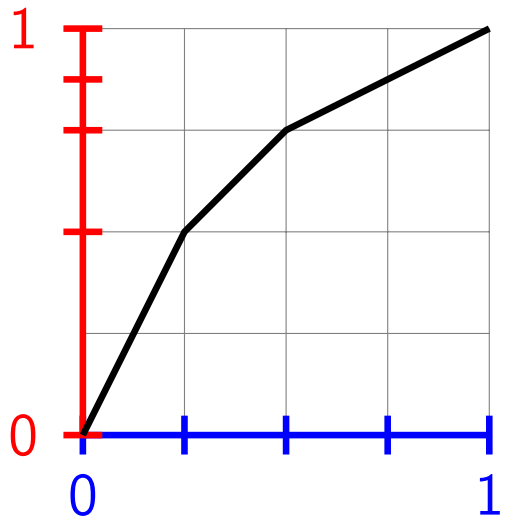
2015 Barker, Duncan and R. Generalisation to $G_{n,r}$ and corrections.
Proof of concept implementation.

Interval partitions $\rightarrow$ trees



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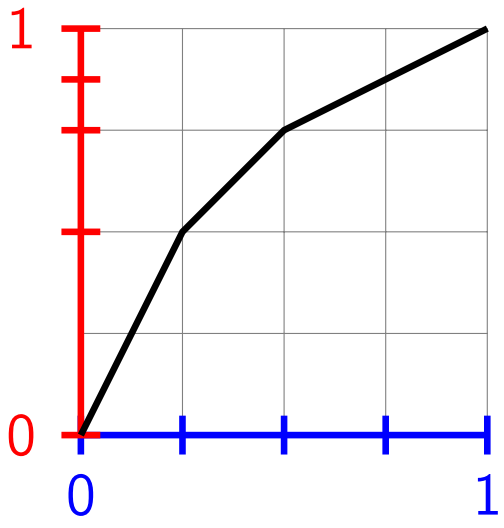

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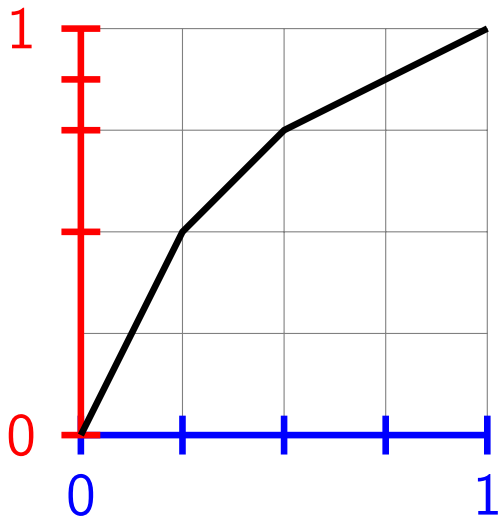

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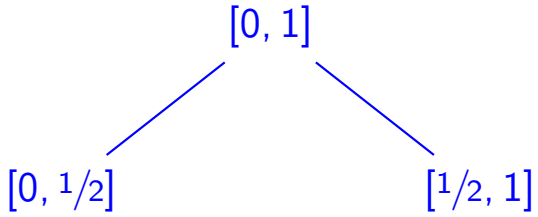

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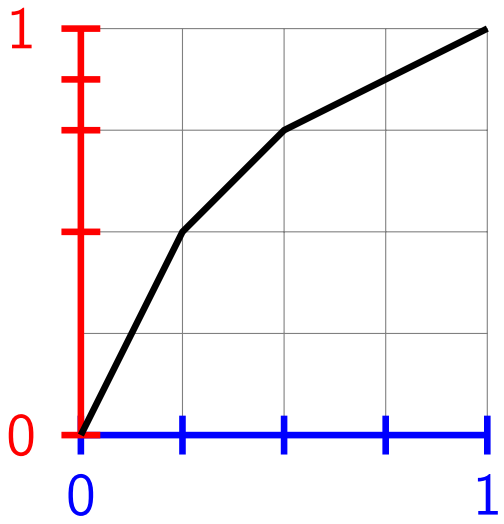

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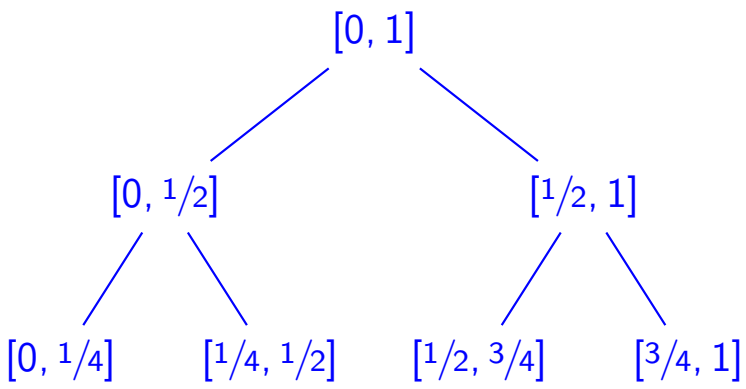

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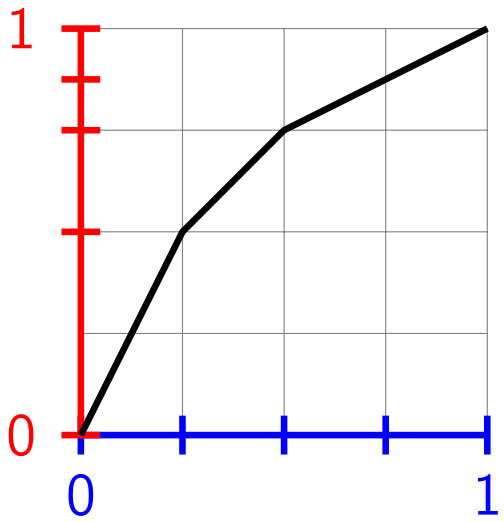

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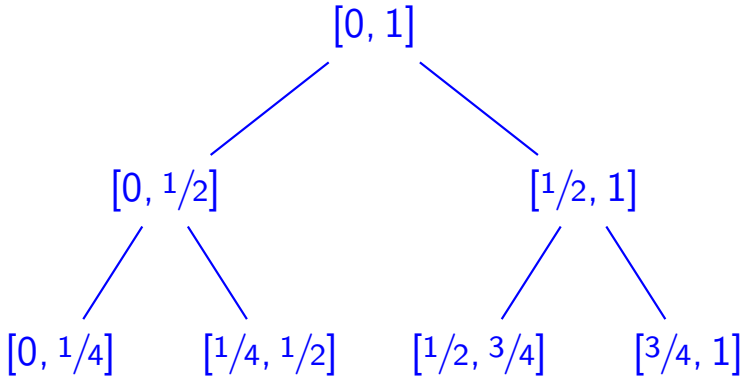

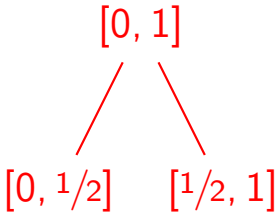

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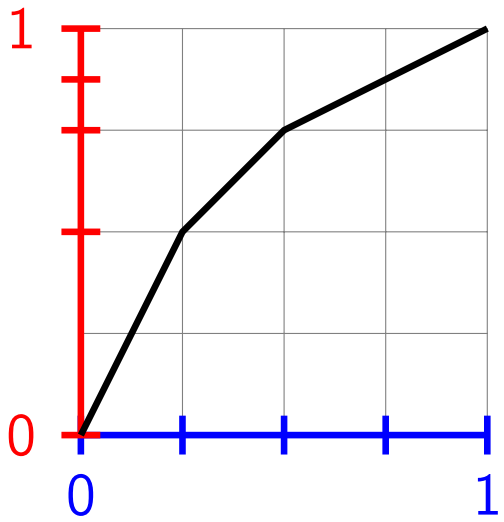

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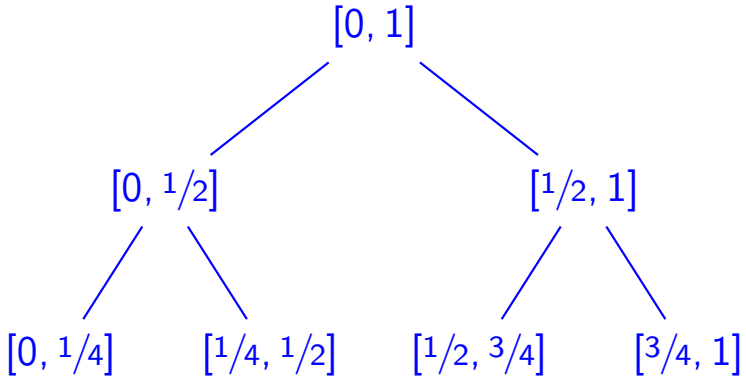

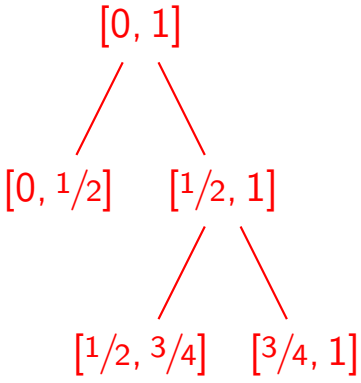

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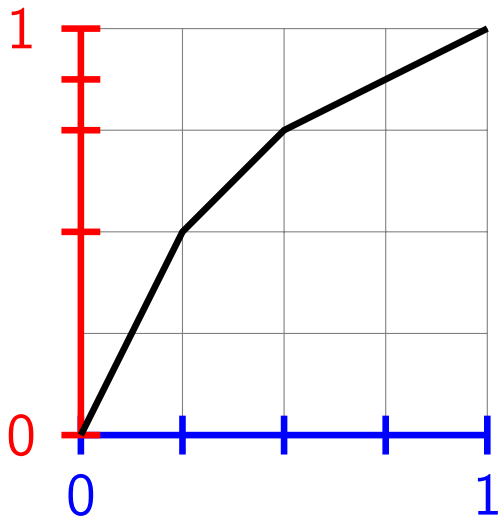

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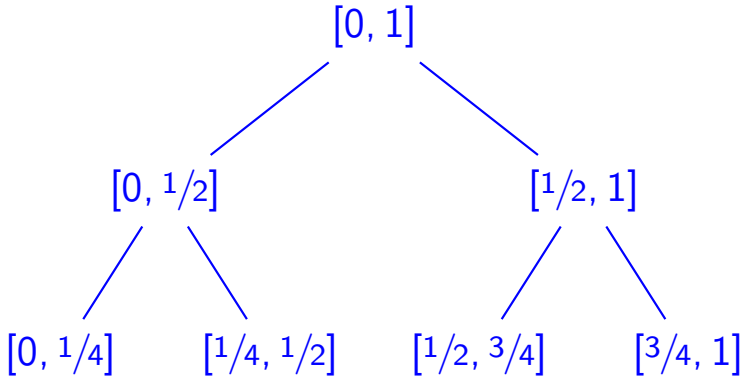

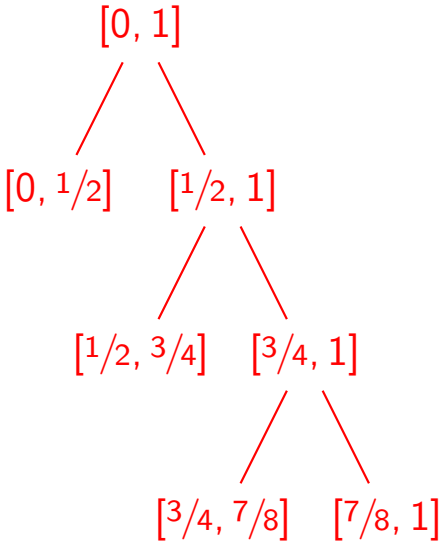

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Higman described paths in the tree using an algebra. Introduce labels:

$$\text{Root} \mapsto x \qquad \text{left} \mapsto \alpha_1 \qquad \text{right} \mapsto \alpha_2$$

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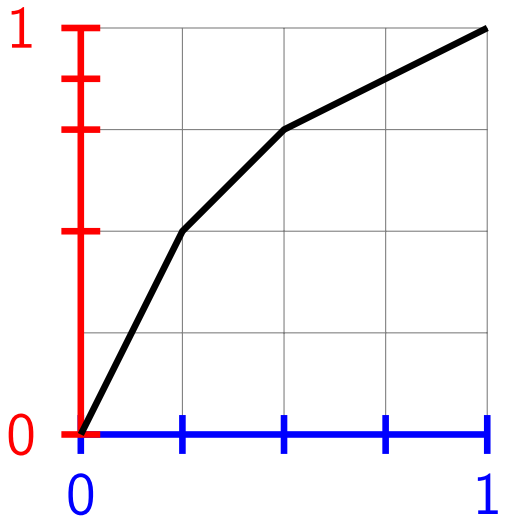

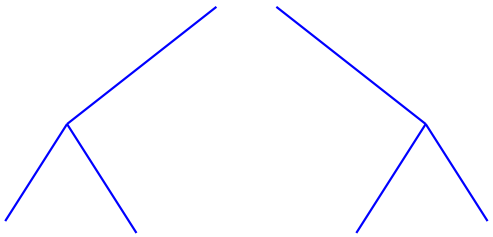

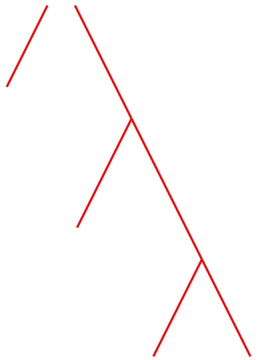

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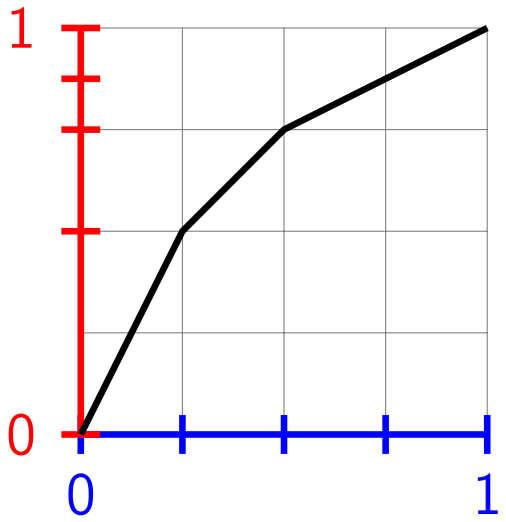

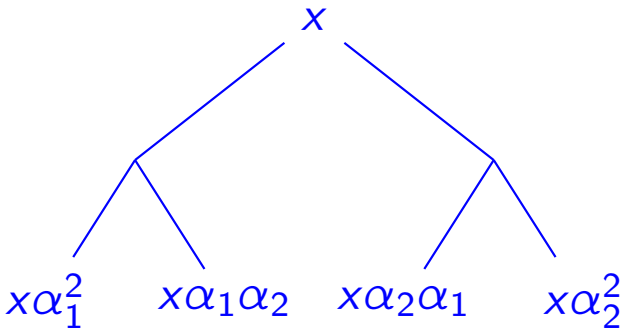

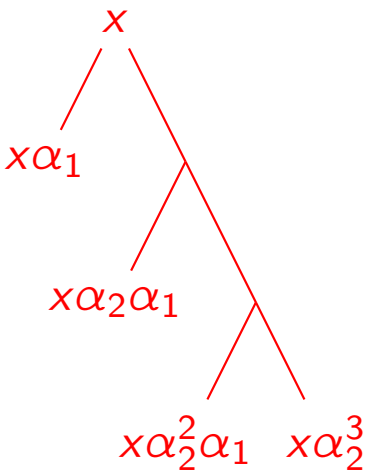

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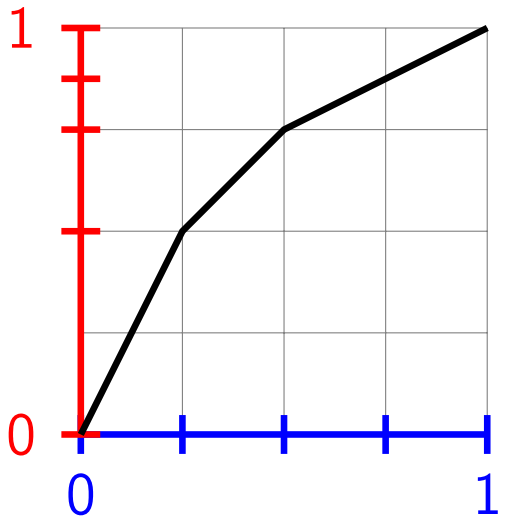
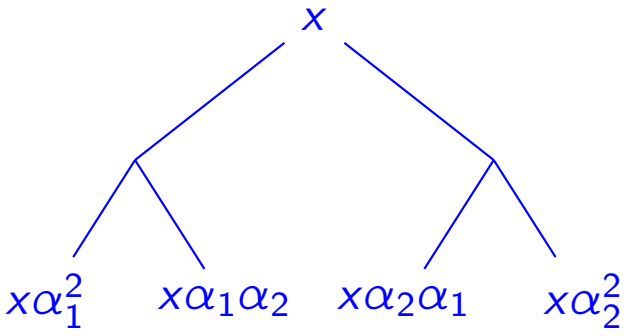
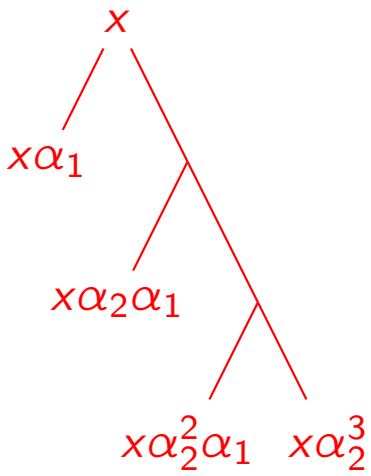


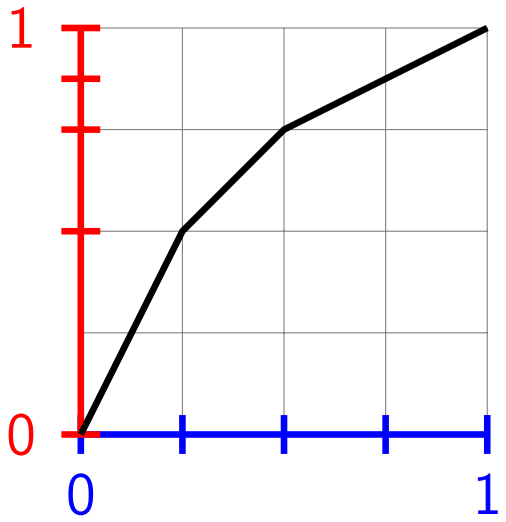


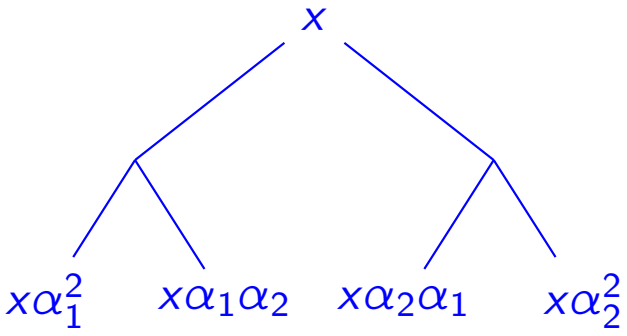


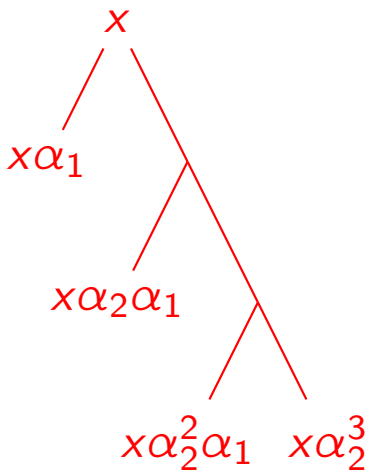
Maps specified by lists of domain and range words.

$$x\alpha_1^2 \mapsto x\alpha_1 \qquad x\alpha_2\alpha_1 \mapsto x\alpha_2^2\alpha_1$$
$$x\alpha_1\alpha_2 \mapsto x\alpha_2\alpha_1 \qquad x\alpha_2^2 \mapsto x\alpha_2^3$$

Trees $\to$ paths and words







Maps specified by lists of domain and range words.

$$\begin{aligned} x\alpha_1^2 &\mapsto x\alpha_1 & \qquad x\alpha_2\alpha_1 &\mapsto x\alpha_2^2\alpha_1 \\ x\alpha_1\alpha_2 &\mapsto x\alpha_2\alpha_1 & \qquad x\alpha_2^2 &\mapsto x\alpha_2^3 \end{aligned}$$

Easier to describe repeated application, e.g.

$$\begin{array}{rcccccc} x\alpha_2^2 & \mapsto & x\alpha_2^3 & \mapsto & x\alpha_2^4 & \mapsto & \dots \\ [3/4, 1] & \mapsto & [7/8, 1] & \mapsto & [15/16, 1] & \mapsto & \dots \end{array}$$

Components ($\approx$ orbits)

- Pick your favourite word e.g. $w = x\alpha_1^2 \leftrightarrow [0, 1/4]$.
- Compute component of w until you can't any more:

$$\ldots,\ f^{-2}(w),\ f^{-1}(w),\ w,\ f(w),\ f^2(w),\ \ldots$$

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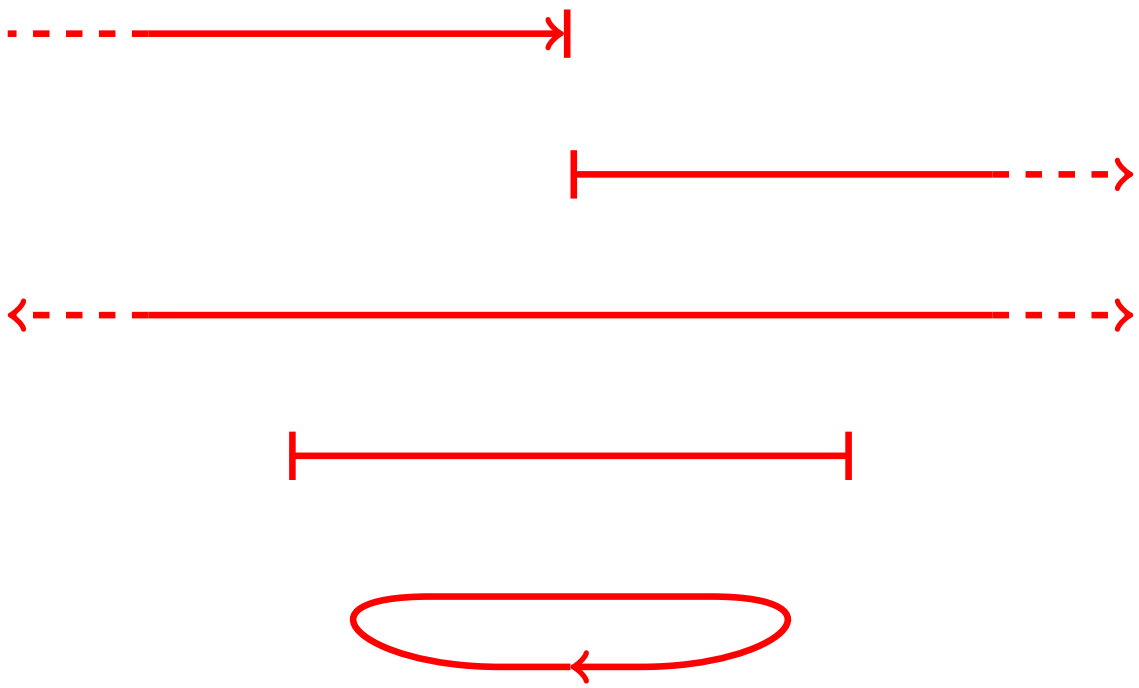
Components come in five different shapes:



Components and Conjugacy

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4. If one works: we get a conjugator h.
5. If none of them work: no conjugator exists.

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- Written in Python: something I knew and quick to work with.
- Summer project that was more like a semester project. . .
- Other tools exist to do calculations in V, but not to solve the conjugacy problem.

Code is on GitHub. Come and find me if you want a demo!

Sphinx: comments in source code

```
def format(word):
    """Turns a sequence of integers representing a *word* into [...]
        >>> format([2, -1, 2, -2, 0])
        'x2 a1 x2 a2 L'
        >>> format([])
        The Spanish Inquisition
    """
    if len(word) == 0:
        return "<the empty word>"
    return " ".join(_char(i) for i in word)
```

Sphinx generates nice HTML documentation and runs tests based on *"""comments like this"""*.

Sphinx: doctest

```
H:\thompsons_v\docs>make doctest
[...]
**********************************************************
File "thompson.word.rst", line 10, in default
Failed example:
    format([])
Expected:
    The Spanish Inquisition
Got:
    ‘<the empty word>’
**********************************************************
1 items had failures:
   1 of 100 in default
100 tests in 1 items.
99 passed and 1 failed.
***Test Failed*** 1 failures.
```

Other lessons learned

- Small test suites—catch bugs before they happen
- Generate random examples
- Immutable words
- Document the code

Future Work

Code

- More testing
- Complexity analysis

Theory

- Simultaneous conjugacy
  Given $x_1, \ldots, x_n; y_1, \ldots, y_n$ find a single conjugator z such that

$$z^{-1} x_i z = y_i, \quad \forall i$$

- Try to solve different kinds of equations?
- Transfer to more general Thompson-like groups $V(\Sigma)$?