A spotter’s guide to fractals

What, Why and How

David Robertson

Wednesday 16th November 2016

WHAT: real world examples

Clouds are not spheres

Mountains are not cones

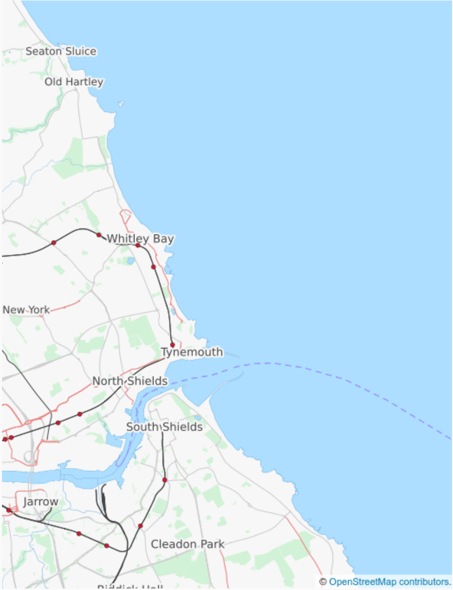
Coastlines are not circles

Bark is not smooth

Lightning doesn’t travel in a straight line

Loads of real-life systems look rough or noisy; can we quantify, model or simulate this?

Hard to describe a coastline



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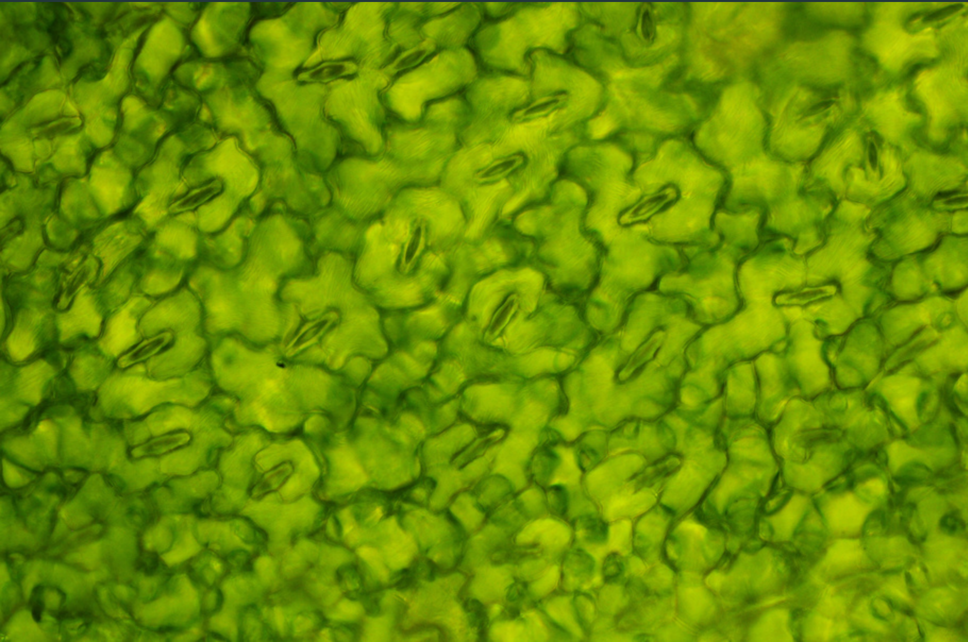

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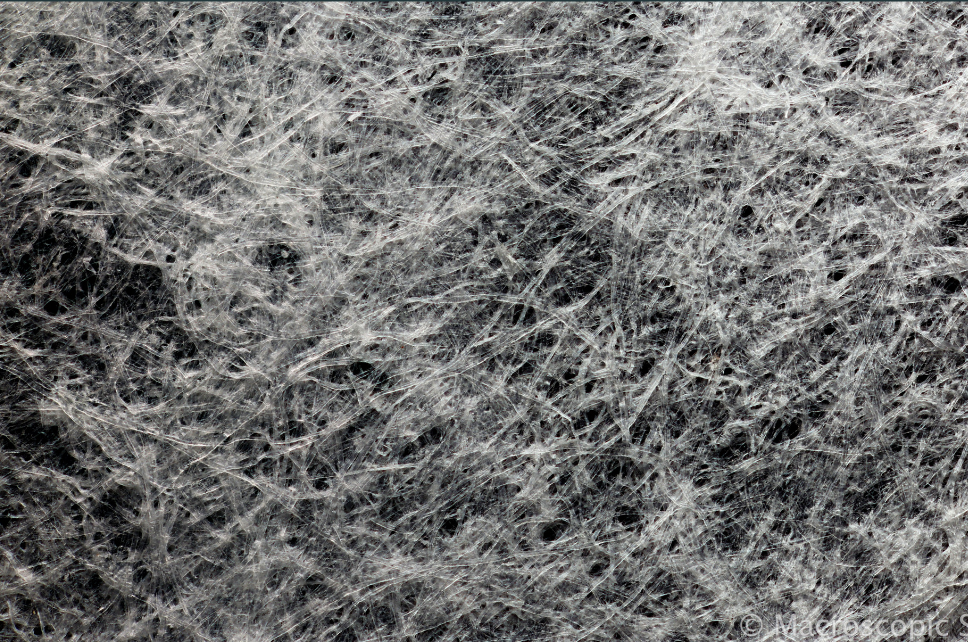

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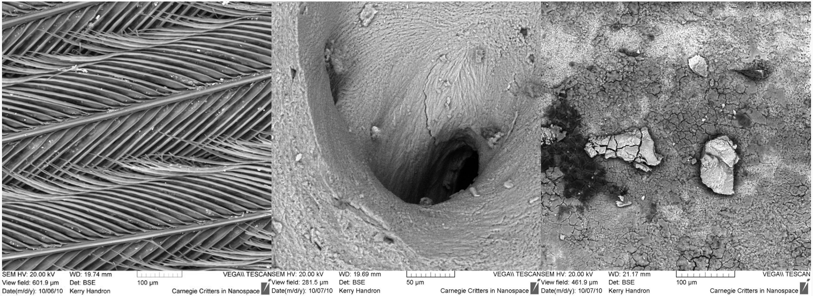

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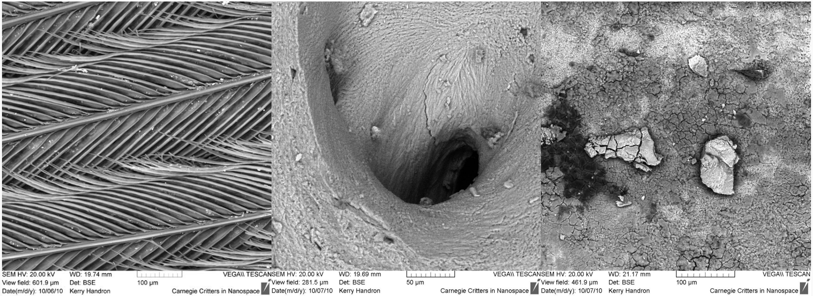

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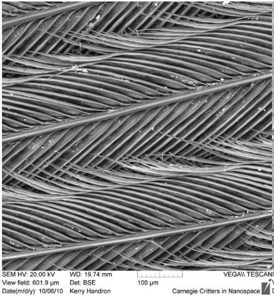

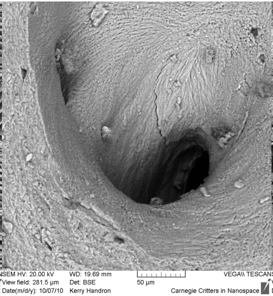

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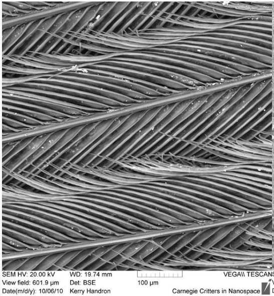

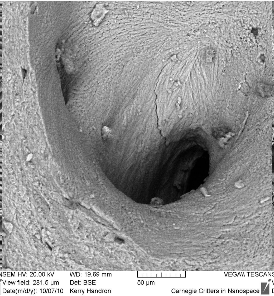

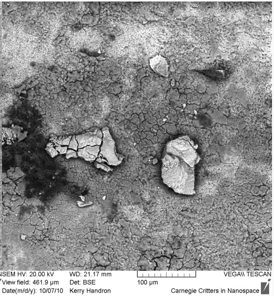

feather bone egg

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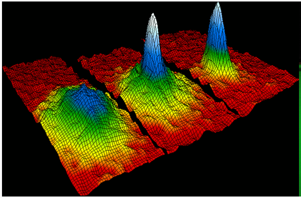
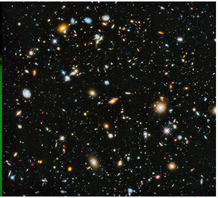
Quantum	Classical	General Relativity

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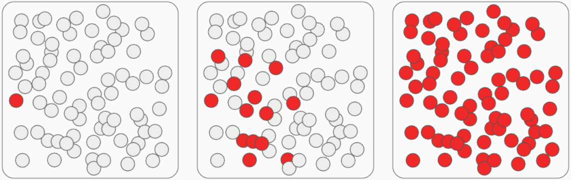

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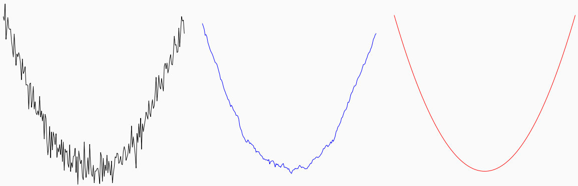
Time series	Moving average	Trend



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Detailed at all scales

WHAT: Mathematical description

A definition

Definition (Mandelbrot)

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Specifying "dimension" turns out to be tricky...

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also *Lebesgue* or *covering* dimension

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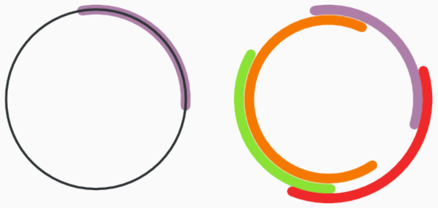

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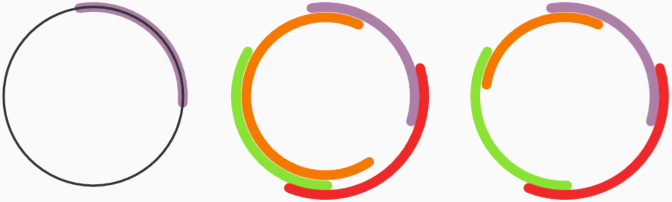

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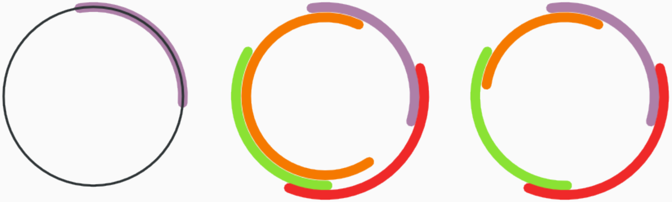

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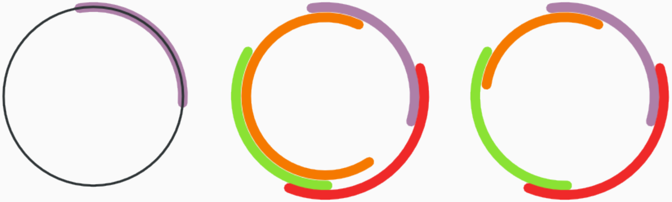

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- The **topological dimension** of X is $\dim_{\mathrm{Top}}(X) = N - 1$.

Box-counting dimension

also *Minkowski* dimension

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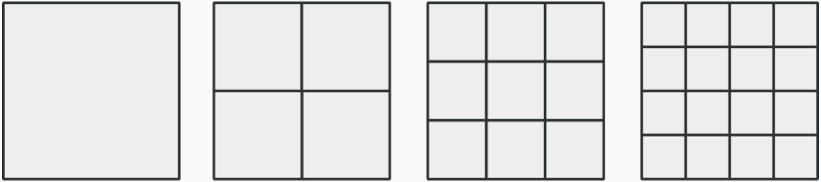

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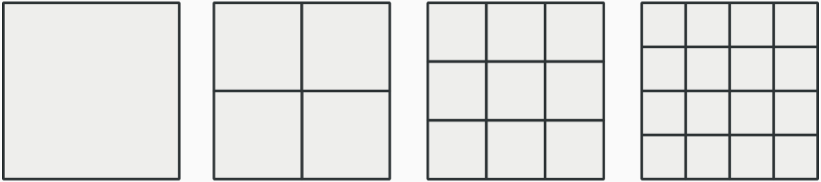

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$$N = (1/n)^2 = r^{-2} \iff \log N = -2\log(r)$$
$$\iff \log N / -\log(r) = 2$$

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Dimension defined by $\dim_{\mathrm{Box}}(X) = \lim_{r \to 0} \dfrac{N(r)}{-\log(r)} \approx 1.25 \notin \mathbb{Z}$!!.

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- information dimension
- correlation dimension
- Assouad dimension
- packing dimension
- ...

HOW: Mathematical models

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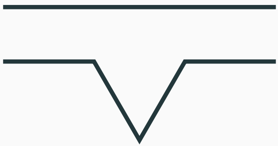

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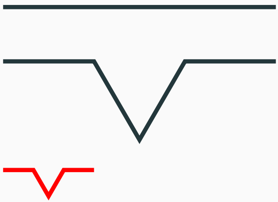

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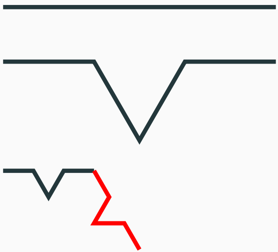

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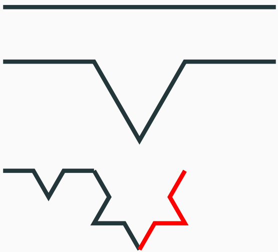

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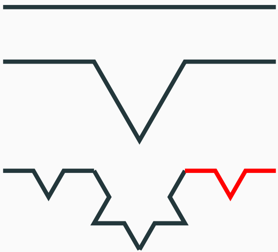

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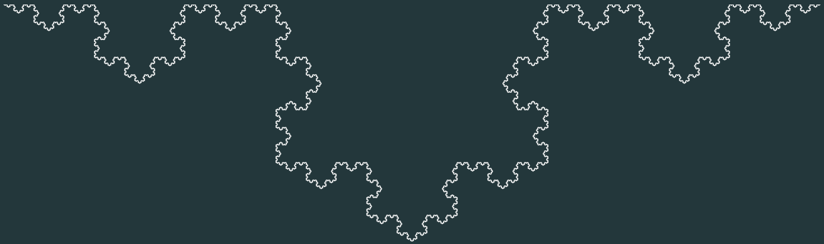

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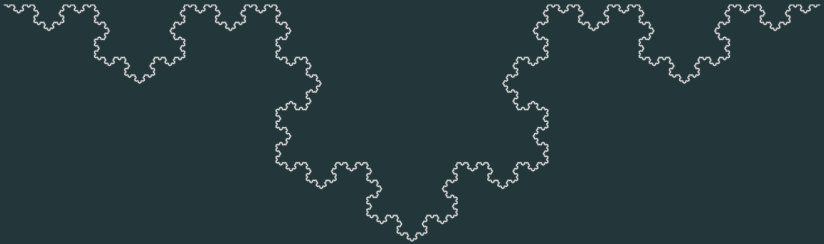

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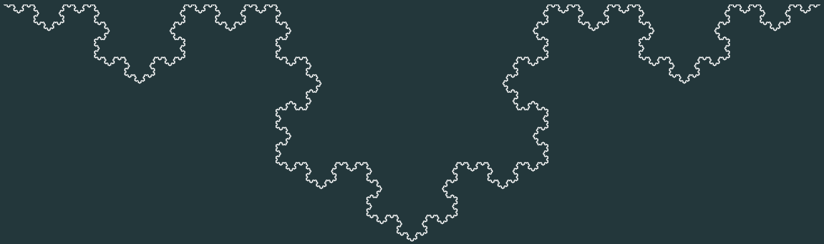

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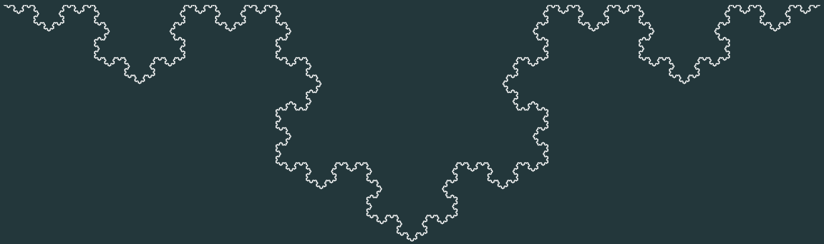

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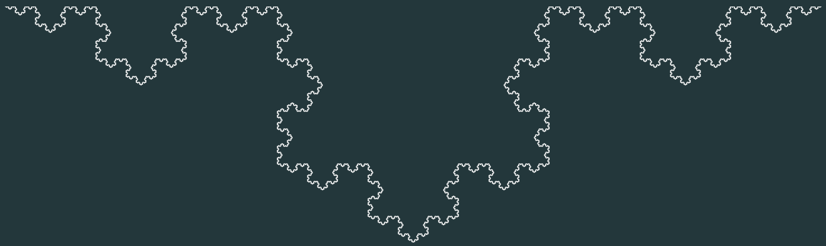

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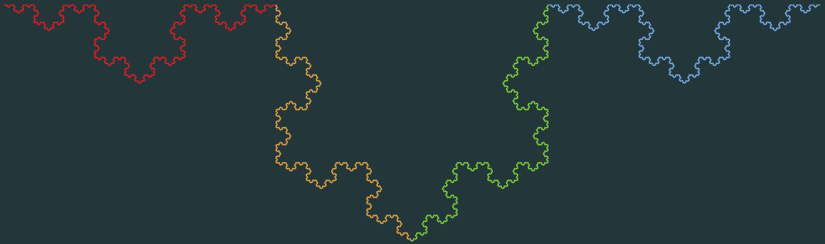
Topological dimension 1

Fractal dimension $\log(4)/\log(3) \approx 1.262$

Similarity dimension



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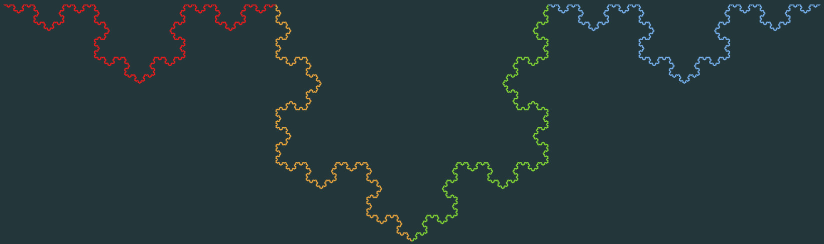

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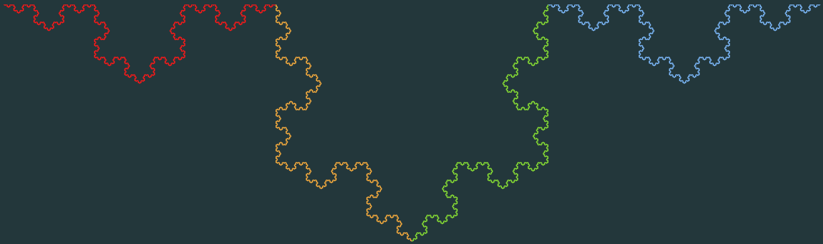

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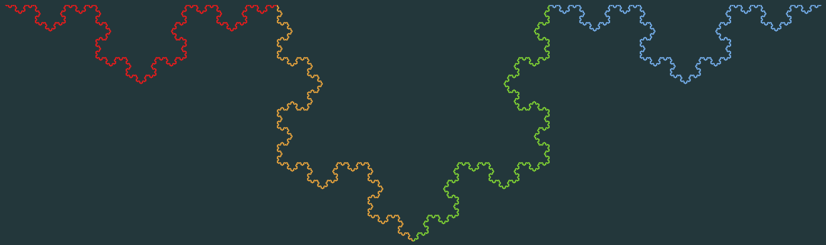

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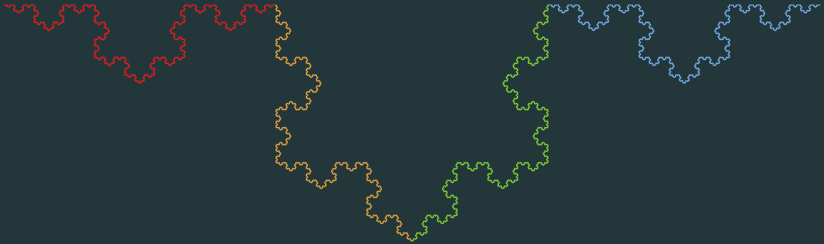

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Webbrowser demo @ `caldew:5000`

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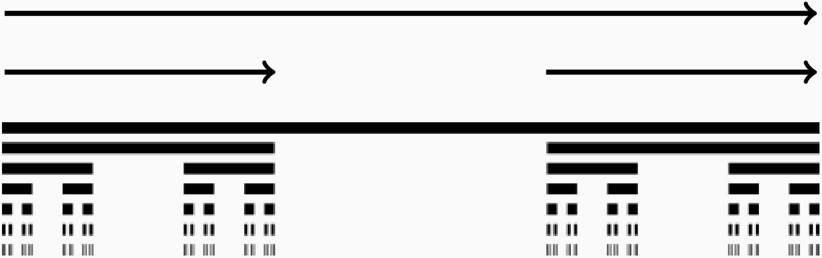

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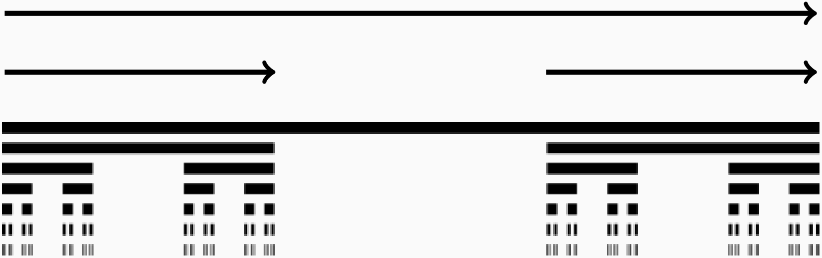

$\dim_{\mathrm{Sim}}(C) = \log 2 / \log 3 \approx 0.6309$ $\qquad$ $\dim_{\mathrm{Top}}(C) = 0$

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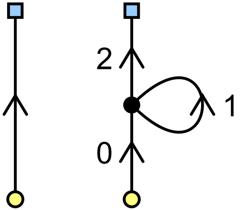
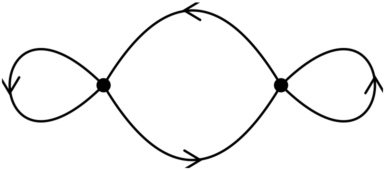
Example: Basilica set $\mathcal{B}$ (an example of a "Julia set")





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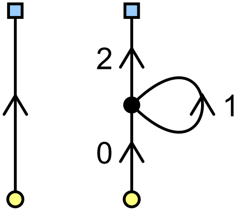

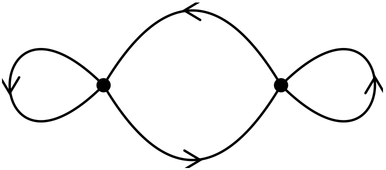

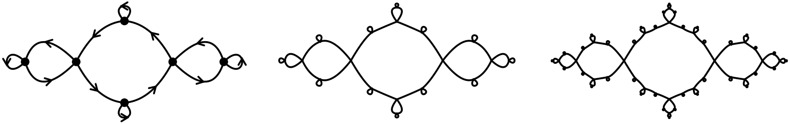

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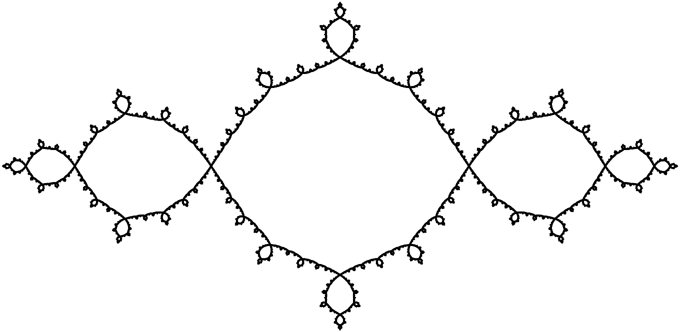

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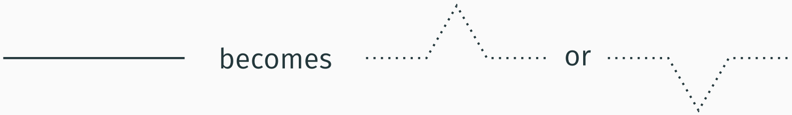

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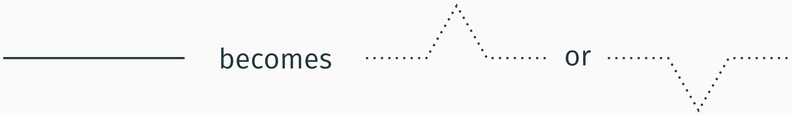
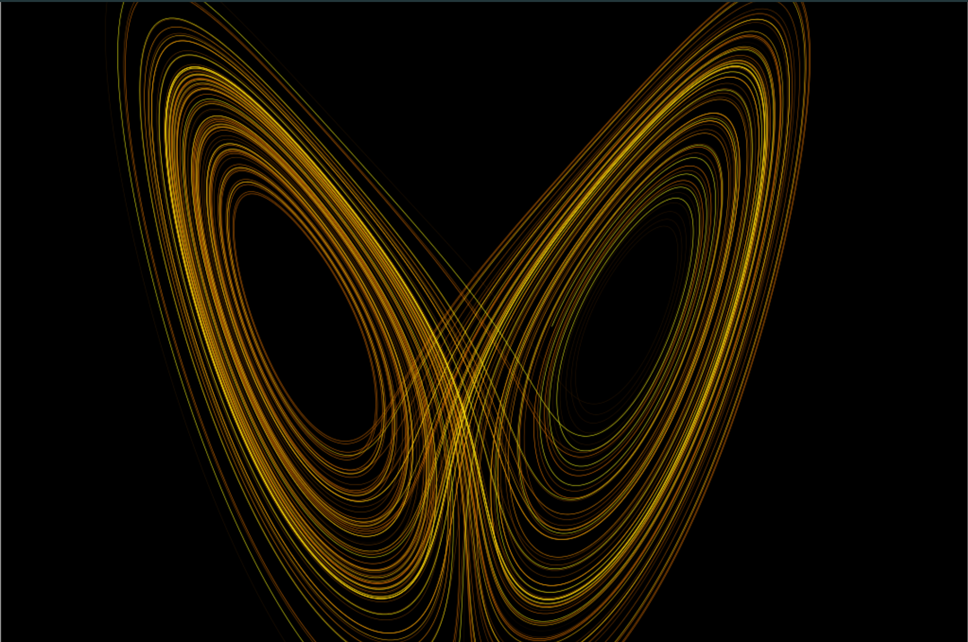


Randomness leads to an entire class of ‘stochastic fractals’:

- Brownian motion
- Self-avoiding walks/paths

Alternatives: L-systems

Alternatives: Strange attractors



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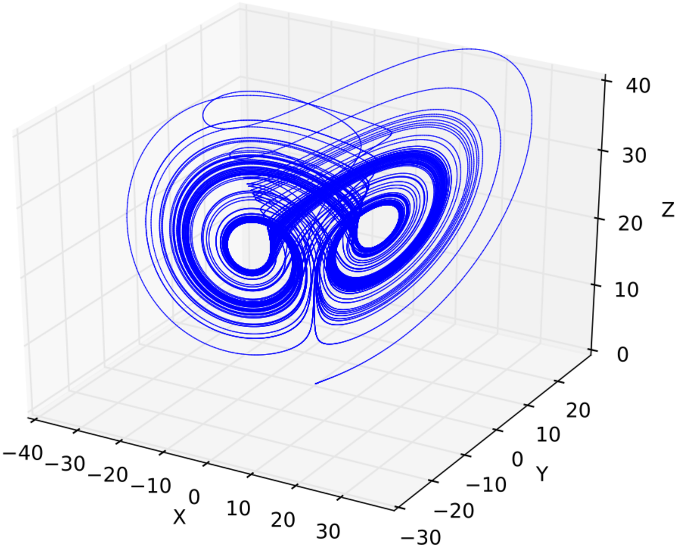
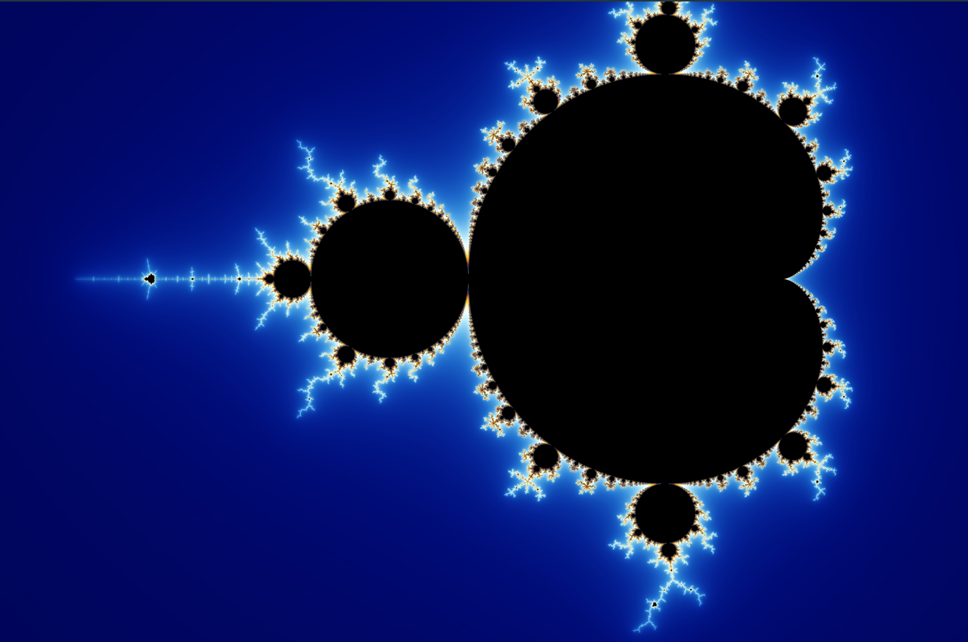


Alternatives: Escape time fractals

Animation



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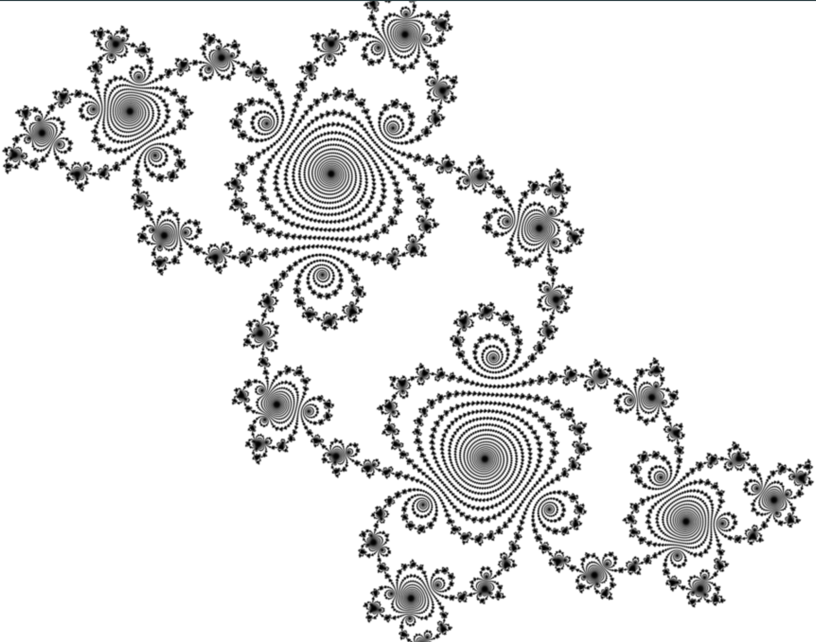

Other ways to make fractals

Iterated function system: copy an object to shrunk versions of itself

Stochastic fractals: detail defined by random movement or deformation

L-systems: based on rewriting strings, good for modelling plants

Strange attractors: points in a chaotic systems often get stuck in a fractal set

Escape time fractals: reapply a map and wait until it sends points to a limit or to ∞

WHY: is it just pretty pictures?

More convincing computer simulations

- Real world is more fractal than not

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- A base for artists to detail, or for further processing

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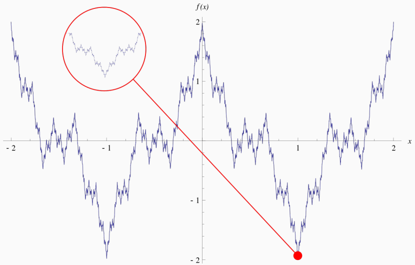
- **Weierstrass function:** $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$



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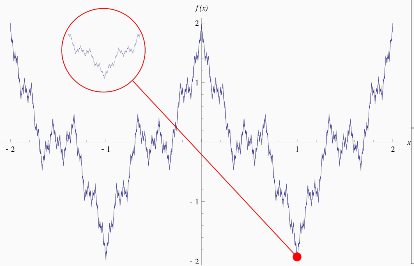
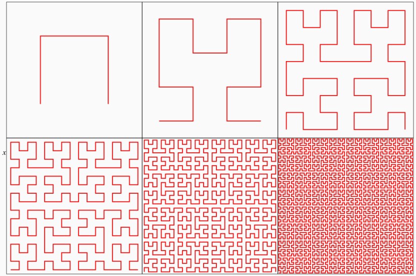
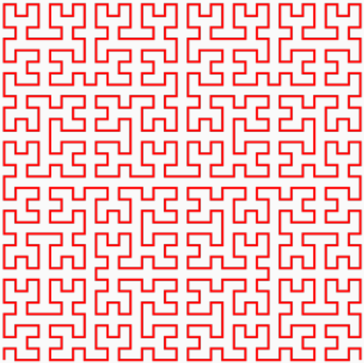
- **Space-filling curves:** continuous map $[0,1] \twoheadrightarrow [0,1]^2$





A form of compression

- Need to walk through a grid with small coordinate changes?



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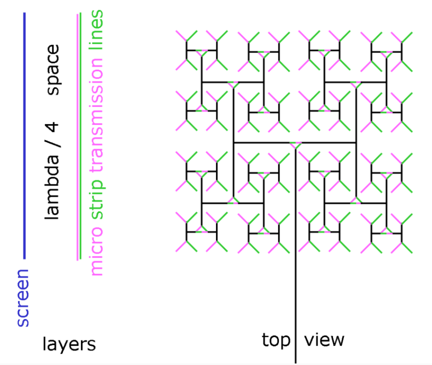

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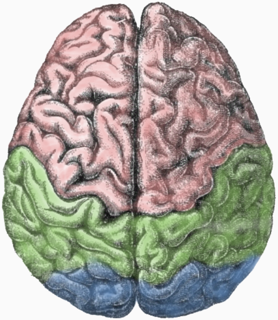

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- Need a systematic way to make a rough surface?

In short, Fractals:

Shapes with built-in self-similarity

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$\implies$ Pretty pictures!

Flickr photos: https://www.flickr.com/photos/ + ...

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 sonofgroucho/5118887516

 qualsiasi/261599589

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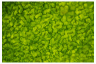
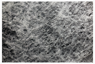
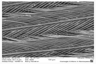
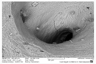
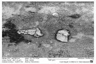
 28594931@N03/4831814540

 107963674@N07/14628897742

 bluegreenchair/5615543558

 bluegreenchair/5614961945

 bluegreenchair/5614962423

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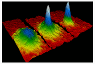
Wikimedia images: https://commons.wikimedia.org/wiki/File: + ...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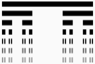
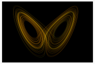
NASA-HS201427a-HubbleUltraDeepField2014-20140603.jpg


Great_Britain_Box.svg


Cantor_set_in_seven_iterations.svg


Fractal-plant.svg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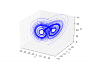
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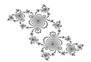
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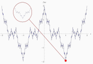
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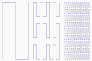
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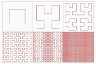
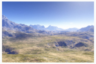

Hilbert_curve.svg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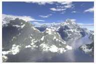
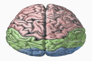
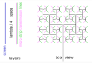

Fractal_terrain_texture.jpg


BlueRidgePastures.jpg


Cerebral_lobes.png


Casts_of_lungs%2C_Marco_resin%2C_1951_(23966574469).jpg


Evening_London_(15884928867).jpg


Antenna_flat_panel.png

Others

- Maps from Open Street Map
- The last three aren't Creative Commons or Public domain:
- YouTube icon from YouTube's branding guidelines
- Basilica images from Belk, Forrest: *Rearrangement Groups of Fractals* @ arXiv:1010.03133
- Fractal sound barrier from

  `http://www.ipam.ucla.edu/research-articles/fractal-acoustic-barrier`

- LaTeX file and source @ GitHub:DMRobertson/fractals

¡Muchas gracias!

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