On centralisers in Thompson's group T

The Thompson family of groups is an infamous collection of finitely-presented gps which provide a series of interesting (counter)-examples in gp theory. They've been nicknamed chameleons because they have a number of different definitions, ~~but~~ and ~~crop up~~ appear in many different contexts.

This talk discusses the middle Thompson group T. We extend work in Mattucci's thesis which exhibits the centraliser of a nontorsion element as a group extension. We show that this extension splits as a direct or wreath product in all but one case; ~~which~~ moreover this case can be detected directly from the element which is to be centralised. rework

In this talk we give a brief introduction to Thompson's gps; motivate ~~this~~ study; summarise Mattucci's extension & give a flavour of our approach to identifying ~~the structure~~ of this ext^n.

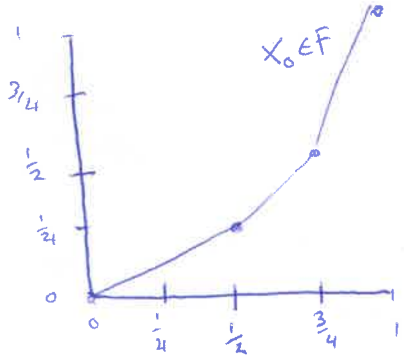
- Introduce gps & motivate (sales pitch)
- Describe & justify Mattucci's result
- Give ~~a flavour of~~ my result & give a flavour of its pf.

Intro

$$F = \left\{ \alpha : [0,1] \to [0,1] \;\middle|\; \begin{array}{l} \alpha \text{ is a piecewise-linear homeomorphism} \\ \text{gradients of } \alpha \in 2^{\mathbb{Z}} = \{2^0, 2^{\pm 1}, 2^{\pm 2}, \ldots\} \\ \text{breakpts of } \alpha \in \mathbb{Z}[\tfrac{1}{2}] = \{ \tfrac{a}{2^n} \mid a, n \in \mathbb{Z} \;\; n \geq 0 \} \end{array} \right\}$$

"dyadic rationals" (dyadics) 1st class citizens

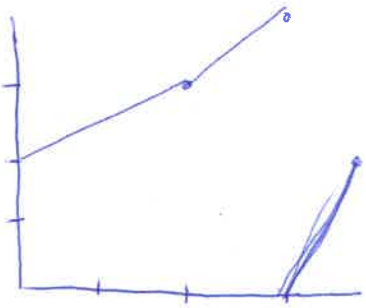
Ex $0, \frac{1}{2}, \frac{3}{8}, \frac{14}{1024}$ $\quad \neg$ Ex $1/3, \frac{2}{15}$



S^1 = circle = $[0,1]/\{0 \sim 1\}$

$T = \{ \alpha : S^1 \to S^1 \mid$ same 3 conditions $\}$

$F = \{ \alpha \in T \mid \alpha(0) = 0 \} \leq T$.



$\alpha \in T \setminus F$

gps under f^n composition.

Motivation : • lots of standard questions are hard to tackle

• algebraically resistant; combinatorial | computationally accessible.

→ crypto system?

Historical: (• F : no nonabelian free subgps. Conjectured nonamenable

• T : among first examples of a f. pres inf. simple gp.

no quotients except $\{1\}$ & T.

Matucci's extension: b/g.

$\alpha \in T$ has centraliser $C_T(\alpha) = \{\beta \in T \mid \alpha\beta = \beta\alpha\}$

(centralisers parameterise conjugacy)

Def ~~α~~ Let α be a circle homeo^M.
The rotation # $\rho(\alpha)$ is defined as the limit

$$\lim_{n\to\infty} \frac{\tilde{\alpha}^n(x) - }{n}$$

for any $x \in S^1$ and lift

Independent of x & $\tilde{\alpha}$

Details not so important

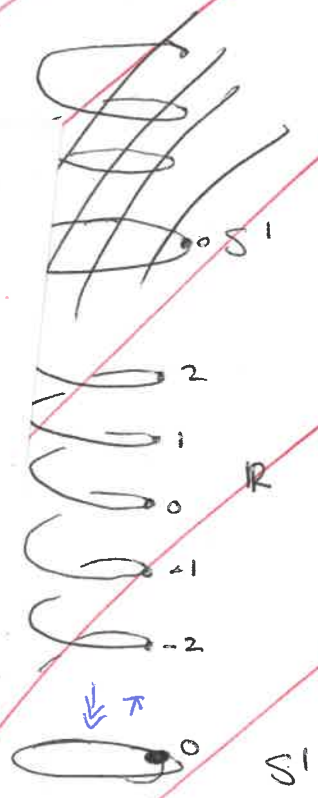
Fact 1: α circle homeo^M, orientating-pres.
If ~~a~~ α has a finite orbit of size q, every fte orbit of α has size q.
(Property of Poincaré rot^n number)

School of Mathematics & Statistics
www.ncl.ac.uk/maths

Fact 2 any $\alpha \in T$ always has a finite orbit, say of size q.
(Brin's revealing pairs for V)

School of Mathematics & Statistics
www.ncl.ac.uk/maths

Poincaré



Properties
- $\rho(\alpha$
- $\rho($
- $\rho(\alpha$
- $\rho(\alpha) = p/q$ (lowest terms) iff α has an orbit of size q, in which case all pdc orbits are of size q.

⚠ $\rho(\cdot)$ is not ~~in general~~ a homomorphism $\text{Homeo}^+(S^1) \to \mathbb{R}/\mathbb{Z}$ — q depends on α

However Def • Remarkable pts $R_\alpha = \partial \text{Fix}(\alpha^q)$ with $\rho(\alpha) = p/q$ reduced. ∂R_α finite

Fact • $\alpha \in T$ nontorsion always has rational $\rho(\alpha)$ & R_α non empty

Fact ~~R_α~~ $C_T(\alpha)$ (acts on) permutes R_α. (justified later)

Thm (Matucci 2008) $\alpha \in T$ nontorsion

$$C_T(\alpha)_0 \hookrightarrow C_T(\alpha) \twoheadrightarrow C_T(\alpha)|_{R_\alpha}$$

Subset of centraliser ~~with f~~ whose elmts have fixed pts ($\cong F^f \times \mathbb{Z}^z$)

(looks like ~~{~~ $C_F(\beta)$)

$\cong \mathbb{Z}_n$ fte cyclic

Sketch justification: LHS = action's kernel, RHS = action's image.

Why do we have an action? That is, why does $C_T(\alpha) \curvearrowright R_\alpha$.

~~$C_T(\alpha) \cdot R_\alpha = C_T(\alpha) \cdot \partial \mathrm{Fix}(\alpha^q)$~~

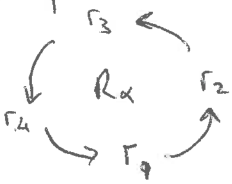
Say $\gamma \in C_T(\alpha)$, i.e. $\gamma\alpha = \alpha\gamma$.

Then $\gamma \cdot R_\alpha = \gamma(\partial \mathrm{Fix}(\alpha^q)) = \partial \gamma(\mathrm{Fix}(\alpha^q)) = \partial \mathrm{Fix}([\alpha^q]^\gamma) = \partial \mathrm{Fix}(\alpha^q) = R_\alpha$.

(∂ commutes w/ homeo^{ms}) ($\beta(\mathrm{Fix}(\alpha)) = \mathrm{Fix}(\alpha^\beta)$)

LHS: in kernel of action $\iff$ fixed ~~#~~ R_α ptwise $\iff$ have rot^n # 0.

RHS: must be a subgp of $\mathfrak{S}(R_\alpha)$ (full perm gp); but also must preserve the cyclic order



$\longrightarrow$ cyclic gp.

(not necc. transitive action)

~~Rmk action need not be transitive~~

Rmks
- effort needed to establish what the kernel "looks like"
- problems if all remarkable pts are nondyadic
- action need not be transitive. (|quotient| < $|R_\alpha|$ possible)

Missing ingredient: How do we reassemble centraliser from the kernel & quotient?

Not obvious:

$$\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}_2 \twoheadrightarrow \mathbb{Z}_2$$
$$\mathbb{Z} \hookrightarrow \mathbb{Z} \rtimes \mathbb{Z}_2 \twoheadrightarrow \mathbb{Z}_2$$
$$\mathbb{Z} \cong 2\mathbb{Z} \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$$

} inequivalent!

Thm. (R) Let $\alpha \in T$ be nontorsion. 4 cases:? Centraliser looks like the following.

	some ~~are all pts in~~ dyadic pts ~~(Yes)~~ in R_α	all nondyadic pts ~~nondyadic? (Yes)~~
$\rho(\alpha) = 0$	$(\mathbb{Z}^z \times F^f) \rtimes_{s_h} \mathbb{Z}_h$ (*)	$\mathbb{Z} \times \mathbb{Z}_h$
$\rho(\alpha) \neq 0$	$(\mathbb{Z}^z \times F^f) \rtimes_s \mathbb{Z}_h$ or nonsplit	$\mathbb{Z} \times \mathbb{Z}_h$

$s \mid h$

(We can identify exactly which of these options can be realised.)

Prop $C_T(\alpha)$ nonsplit iff α in that box and ~~α has no q^{th} root~~ $\nexists$ q^{th} root ω st $\omega^q = \alpha$ with ω has fixed pt

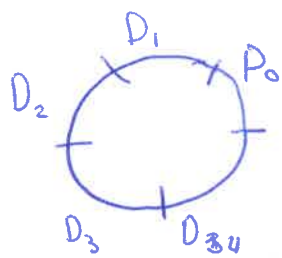
Reminder/Not^n $A \wr_S B$ means $A^S \rtimes B$ where ~~B~~
conjugating by B shuffles the copies of A around. s|h

Proof Lots of cases, details to check & calculations.
Give the strategy & a flavour here.

Plan for each of the four cases:

1. Define a "block form": recipe for building elmts which belong to this case.
2. Find a way to ~~express~~ write down centralising elmts in terms of the block form
3. Check that every ~~ex~~ elmt in this case has a maximal block form (make sure we've got all the centraliser)
4. Multiply centralising elmts to see how they combine; infer centraliser structure from this.

Ex easiest case ⊛



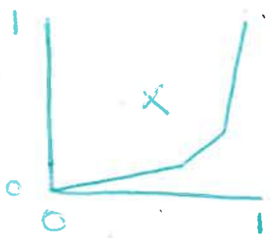
1. **Def** Block form: • circular partition $D_0, \dots, D_{h-1}$
∂D_i dyadic

• maps $\Sigma_i : D_i \to [0,1]$ ~~with~~ wiry
  • PL, gradients $2^{\mathbb{Z}}$, dyadic breakpts

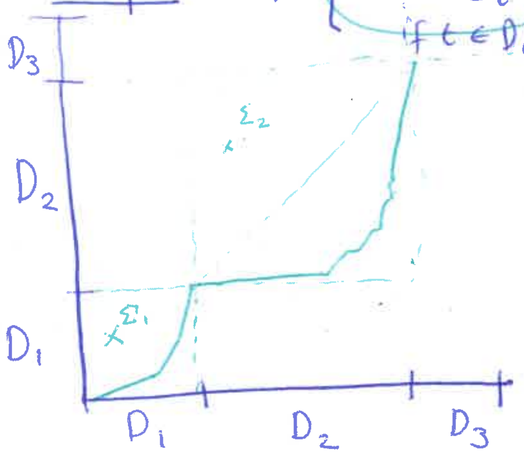
• $x \in F \setminus \{id\}$.

$(x; \Sigma_1, \dots, \Sigma_h)$ has h blocks

maximal if $\nexists$ block form with $h' > h$ blocks.



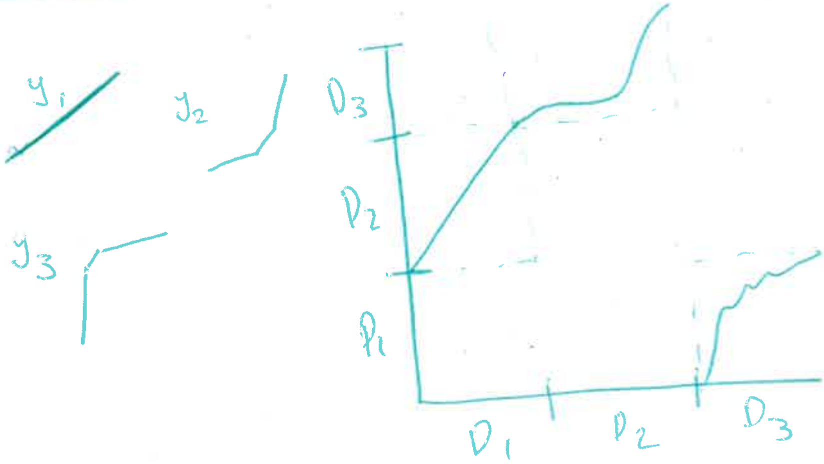
Recipe $\alpha(t) = \begin{cases} \Sigma_i^{-1} \circ \alpha \circ \Sigma_i(t) \\ \text{if } t \in D_i \end{cases}$



Σ_i stretch/squash
& conjugate in F

2. Define an element θ in terms of $y_1, \dots, y_h \in F$ and an index $0 \le d < h$.

Recipe: $\theta(t) = \{ \Sigma_i^{-1} y_i \Sigma_{i+d}(t)$ if $t \in D_i$



Can check $\theta\alpha = \alpha\theta$

$(\alpha\theta)|_{D_i} = \alpha|_{D_{i+d}} \theta|_{D_i} = \Sigma_{i+d} x \Sigma_{i+d}^{-1} \Sigma_{i+d} y_i \Sigma_i^{-1}$

$= \Sigma_{i+d} \, x \, y_i \, \Sigma_i^{-1}$

$\Vert$ cos $y_i x = x y_i$

$\theta\alpha|_{D_i} = \theta|_{D_i} \alpha|_{D_i} = \Sigma_{i+d} \, y_i \, \Sigma_i^{-1} \Sigma_i \, x \, \Sigma_i^{-1}$

3: Existence of max block forms

Technical details omitted.

Ideas:
- D_i delimited by remarkable pts
- Find out the 'smallest shift' of rmkble pts
- $x = \alpha|_{D_1}$, any $\Sigma_2, \dots \Sigma_h$

4 Centraliser elmt interaction

$C_F(x)^h \longleftrightarrow (y_1, \dots, y_h)$

$\{K\} \hookrightarrow C_T(\alpha) \twoheadrightarrow \mathbb{Z}_h$, $\langle\delta\rangle$

Kernel elmts: $d = 0$ $K|_{D_i} = \Sigma_i^{-1} y_i \Sigma_i$

Quotient elmts: F params $= 0$ $\delta|_{D_i} = \Sigma_{i+1} \Sigma_i^{-1}$

$\delta^h = \Sigma_{i+h} \Sigma_{i+h-1}^{-1} \cdots \Sigma_{i+1} \Sigma_i^{-1} = \Sigma_{i+h} \Sigma_i^{-1} = \Sigma_i^{-1} \Sigma_i = id$

$(K^\delta)|_{D_i} = \delta|_{D_{i-1}} K|_{D_{i-1}} \delta^{-1}|_{D_i}$

$= \Sigma_i \, \Sigma_{i-1}^{-1} \Sigma_{i-1} \, y_{i-1} \, \Sigma_{i-1}^{-1} \Sigma_{i-1} \, \Sigma_i^{-1}$

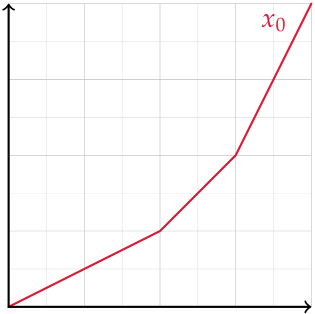
$= \Sigma_i \, y_{i-1} \, \Sigma_i^{-1}$

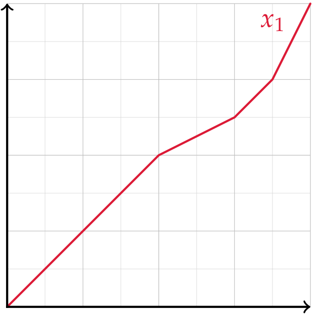
So $K^\delta \longleftrightarrow (y_h, y_1, y_2, \dots, y_{h-1})$

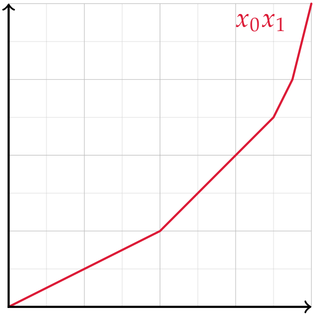
$C_F(x)^h \rtimes \mathbb{Z}_h$

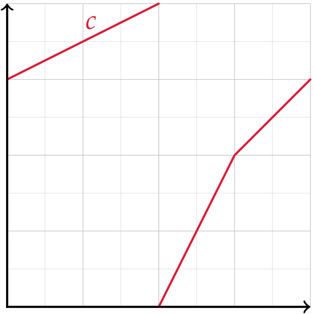
$C_F(x) \wr \mathbb{Z}_h$.

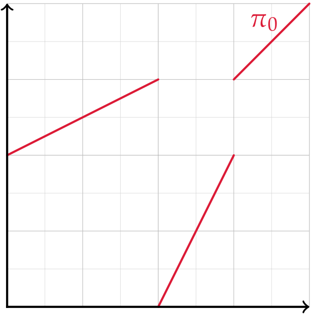
end

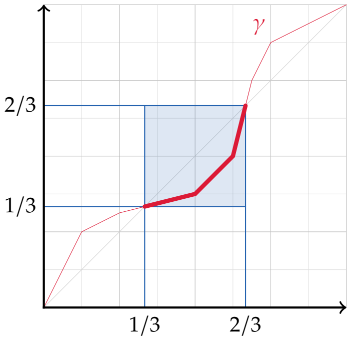

x_0

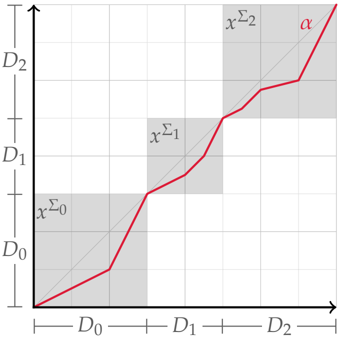

x_1

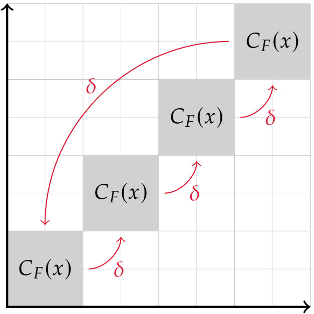

x_0x_1


c

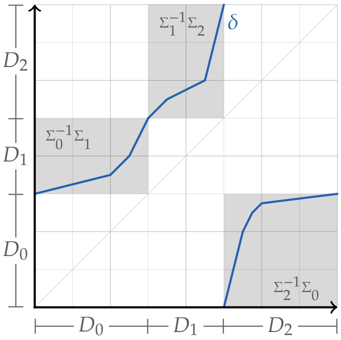

π_0


γ
2/3
1/3
1/3
2/3


x^{Σ_2}
α
x^{Σ_1}
x^{Σ_0}
D_2
D_1
D_0
D_0
D_1
D_2



$C_F(x)$
$C_F(x)$
$C_F(x)$
$C_F(x)$
δ
δ
δ
δ
δ



D_2
D_1
D_0
$\Sigma_1^{-1}\Sigma_2$
δ
$\Sigma_0^{-1}\Sigma_1$
$\Sigma_2^{-1}\Sigma_0$
D_0
D_1
D_2